Optical Signatures of a Tightly Confined Bose Condensate

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Optical properties of a Bose condensate of N_c atoms tightly confined to a harmonic oscillator potential are studied. A resonantly illuminated condensate scatters as much light as x independent atoms, where x depends on the size of the condensate and on the wavelength of light but not on N_c; the linewidth of the resonance is N_c/x times larger than the linewidth of an individual constituent atom. If the condensate is bigger than a wavelength, light is predominantly scattered in the forward direction.

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Laser cooling of atoms to microkelvin temperatures and even below [1-3], substantial atomic densities reached in various traps [4,5], and cooling methods that do not belong to the traditional domain of laser cooling [6,7] have bolstered the hopes that Bose condensation of a weakly interacting gas will be achieved. Considering the prominence of lasers in such schemes, the question of the optical signatures of the Bose condensate immediately arises. In fact, it has been predicted that a spatially homogeneous dense condensate reflects off all near-resonant light [8,9].

In this Letter we discuss induced and spontaneous optical processes in a condensate of an ideal Bose gas *bound by an external potential.* Response of a bound condensate to short light pulses was discussed by Lewenstein and You [10], but here the focus is on steady-state excitation. Diametrically opposite to the earlier results [8,9], we find that on resonance light scattering from the condensate is strongly suppressed in comparison with scattering from the same number of noncooperating atoms. The difference arises because in our examples the condensate is small enough that conservation of momentum is somewhat relaxed. We refer to this circumstance as tight confinement. The resonance line of the condensate is enormously broad, which may be of help in the detection of Bose condensation. You, Lewenstein, and Cooper [11] have recently come to the same conclusion using more elaborate methods.

The effects of the Bose-Einstein statistics depend on the occupation numbers of the individual quantum states. A fully quantized treatment of the center-of-mass (c.m.) motion of the atoms is therefore mandatory. In our simplified model atoms with mass M move in an isotropic harmonic oscillator (HO) potential characterized by the mechanical oscillation frequency ν. The length and momentum scales of the HO are denoted by $l=(\hbar/M\nu)^{1/2}$ and $\hbar/l$, respectively. The eigenstates of the c.m. motion $|\mathbf{n}\rangle$ are labeled by the index $\mathbf{n}\equiv(n_x,n_y,n_z)$, and the corresponding eigenfrequencies are called $\varepsilon_{\mathbf{n}}$. The index $\mathbf{n}=0$ stands for the lowest-energy c.m. state. For the internal states of the atom we adopt the conventional two-state model: ground state $|g\rangle$, excited state $|e\rangle$. These are separated by the frequency ω, and the dipole moment matrix element between the states is called $\mathbf{d}$. The joint internal and c.m. states of the atoms are denoted as $|g\mathbf{n}\rangle\equiv|g\rangle|\mathbf{n}\rangle$, etc. Finally, in the most general case the electromagnetic field is quantized. We enumerate the plane wave modes of the field by an index q that incorporates both the wave vector of the photon $\mathbf{q}$ and the polarization vector $\mathbf{e}_q$. The frequency of the mode q is $\Omega_q=c|\mathbf{q}|$.

Our development involves several parameters, limits, and conditions. We therefore pause to discuss representative numerical values, which may be plugged in later to check that the assumptions of the theory are plausible. However, it should be understood that our intent is *not* to anticipate or design a specific experiment. We use the D_2 line optical transitions of Cs atoms in a trap with the oscillation frequency $\nu=2\pi\times10$ Hz [4] as our example. We estimate the temperature for Bose condensation as $T=0.1$ μK. The maximum number of atoms in the normal fraction, and hence the minimum number of atoms needed to obtain Bose condensation, scales like $N_n=1.202(k_BT/\hbar\nu)^3$ [12], giving $N_n=10^7$. If the total number of atoms N exceeds N_n, the excess $N_c=N-N_n$ atoms go to the ground state $|g0\rangle$ and form the Bose condensate. Let us say half of the atoms are in the condensate; then $N_c=10^7$. The HO length scale, and hence the characteristic size of the condensate, is $l=3$ μm, and for resonant light we obtain $l|\mathbf{q}|=20$. The matrix element $\mathbf{d}$ is taken such that the correct optical linewidth $\gamma=d^2\omega^3/6\pi\hbar\varepsilon_0c^3=2\pi\times2.6$ MHz results. The gas is homogeneously broadened; at 0.1 μK the frequencies of the c.m. motion of thermal atoms and the linewidth are bound to satisfy $\varepsilon_{\mathbf{n}}\ll\gamma$.

We borrow a few standard procedures from the theory of light pressure [13] and from many-body theory [14]. For the time being we assume a truly ideal (noninteracting) Bose gas. The Hamiltonian for the system consisting of the Bose gas and the photons thus is

$$\frac{H}{\hbar}=\sum_{\mathbf{n}}[\varepsilon_{\mathbf{n}}b^\dagger_{g\mathbf{n}}b_{g\mathbf{n}}+(\varepsilon_{\mathbf{n}}+\omega)b^\dagger_{e\mathbf{n}}b_{e\mathbf{n}}]+\sum_q\omega_qa^\dagger_qa_q-\sum_{\mathbf{n},\mathbf{n}',q}[\xi(q)\langle\mathbf{n}'|e^{i\mathbf{q}\cdot\hat{\mathbf{r}}}|\mathbf{n}\rangle b^\dagger_{e\mathbf{n}'}b_{g\mathbf{n}}a_q+\text{H.c.}]\,. \quad (1)$$

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The first two terms in (1) are the energies of the ground-state and excited-state atoms, and the third term is the Hamiltonian of the free photon field. The final term governs the atom-field interactions. For instance, a process in which absorption of a photon in the state q converts a ground-state atom in the c.m. state $|\mathbf{n}\rangle$ to an excited atom in the c.m. state $|\mathbf{n}'\rangle, |g\mathbf{n}\rangle \to |e\mathbf{n}'\rangle$, is governed by the matrix element $-\xi(q)\langle\mathbf{n}'|e^{i\mathbf{q}\cdot\hat{\mathbf{r}}}|\mathbf{n}\rangle$. The coupling coefficient pertaining to the internal states is $\xi(q) = \sqrt{\Omega_q/2\hbar\varepsilon_0 V}\,\mathbf{e}_q\cdot\mathbf{d}$, where V is the quantization volume. The matrix element between the motional states $|\mathbf{n}\rangle$ and $|\mathbf{n}'\rangle$ derives from the c.m. position operator $\hat{\mathbf{r}}$. $e^{i\mathbf{q}\cdot\hat{\mathbf{r}}}$ is a unitary momentum translation operator for the center of mass. The explicit matrix element in (1) thus is the overlap between the c.m. state $|\mathbf{n}'\rangle$ and the c.m. state $|\mathbf{n}\rangle$ shifted by the momentum $\hbar\mathbf{q}$. The condensate is modeled in the customary way by treating its annihilation operator b_{g0} as a c number,

$$b_{g0} \equiv \sqrt{N_c}\,. \tag{2}$$

We assume that the condensate is probed by a weak laser beam, whose mode index and frequency are denoted by k and Ω. The response reflects a balance between transitions driven by the external field, and spontaneous-emission damping that gives a finite lifetime to the excited atomic states. The spontaneous widths of the excited state are much larger than any relevant c.m. frequency $\varepsilon_\mathbf{n}$. Consequently, we proceed from now on as if $\varepsilon_\mathbf{n} \equiv 0$.

As usual, the external field can be regarded as classical, and the corresponding photon operator can be treated as a c number. The part of the Hamiltonian responsible for induced excitations may be rewritten

$$\frac{H}{\hbar} = \cdots - \sum_{\mathbf{n},\mathbf{n}'} [\kappa e^{-i\Omega t}\langle\mathbf{n}'|e^{i\mathbf{k}\cdot\hat{\mathbf{r}}}|\mathbf{n}\rangle b^\dagger_{e\mathbf{n}'} b_{g\mathbf{n}} + \text{H.c.}]\,, \tag{3}$$

where $\kappa = \mathbf{d}\cdot\mathbf{E}/2\hbar$ is the Rabi frequency that ensues when a classical field with amplitude $\mathbf{E}$ drives the internal transition in a single atom. This classical-field Hamiltonian contains a subtle assumption: The driving field is of the undamped and dispersionless form $e^{i\mathbf{k}\cdot\mathbf{r}}$ in the whole sample. In effect, we take the condensate to be optically thin. We return to this assumption below.

Equation (3) suggests that one should resort to the boson operators

$$b^\dagger_\mathbf{n} = \sum_{\mathbf{n}'} \langle\mathbf{n}'|e^{i\mathbf{k}\cdot\hat{\mathbf{r}}}|\mathbf{n}\rangle b^\dagger_{e\mathbf{n}'} \tag{4}$$

of the excited states $|e\psi_\mathbf{n}\rangle$ obtained with the momentum translation $\hbar\mathbf{k}$ from the original excited states $|e\mathbf{n}\rangle$. The driving field then only couples states in pairs $\{|g\mathbf{n}\rangle, |e\psi_\mathbf{n}\rangle\}$. Also, in line with Eq. (2), b_{g0} is viewed as the c number $\sqrt{N_c}$. All matrix elements referring to the state $|g0\rangle$ therefore pick up a large multiplier $\sqrt{N_c}$. Combining these two observations, we retain in our theory only the condensate and its corresponding excited state $|e\psi_0\rangle$. The Hamiltonian becomes

$$\frac{H}{\hbar} = \omega b_0^\dagger b_0 + \sum_q \Omega_q a_q^\dagger a_q - \kappa\sqrt{N_c}(e^{-i\Omega t} b_0^\dagger + \text{H.c.}) - \sqrt{N_c}\sum_q [\xi(q)\langle 0|e^{i(\mathbf{q}-\mathbf{k})\cdot\hat{\mathbf{r}}}|0\rangle b_0^\dagger a_q + \text{H.c.}]\,. \tag{5}$$

The Hamiltonian (5) describes the familiar problem of a quantum HO with frequency ω, driven by a classical force at the frequency Ω and coupled to a photon bath [15]. The only twist is the matrix element $\langle 0|e^{i(\mathbf{q}-\mathbf{k})\cdot\hat{\mathbf{r}}}|0\rangle$. To consider this we continue to assume that the wavelength of the driving light is smaller than the size of the condensate, $l|\mathbf{k}| \gg 1$. The largest momenta available in the HO ground state $|0\rangle$ are of the order $\hbar/l$, so the momentum in the excited state $|e\psi_0\rangle$ is also spread by $\sim\hbar/l$. Spontaneous emission from $|e\psi_0\rangle$ back to $|g0\rangle$ is possible only for photon modes q whose momenta $\hbar\mathbf{q}$ roughly lie in a sphere of radius $\hbar/l$ around the momentum of the incident photons $\hbar\mathbf{k}$. All of this is expressed in the matrix element.

By implication, the range of frequencies of photons that couple to the atomic excitations is restricted too: $\Delta\omega \sim c/l$. Nonetheless, we take $\Delta\omega$ to be much larger than any other relevant frequency parameter in the problem, and thus assume that the coupling is effectively independent of the frequency of the photon. This "flat continuum" case permits the Markov approximation as usual [15], with the usual result that the coupling to the photon bath leads to exponential damping of the HO.

Momentum conservation only permits spontaneous photons whose momenta lie in a cone with opening angle $\sim(l|\mathbf{k}|)^{-1}$ around the momenta of the incident photons $\hbar\mathbf{k}$. The rate of emission is thereby reduced by a factor $\propto(l|\mathbf{k}|)^{-2}$ corresponding to the solid angle available for spontaneous photons. On the other hand, the explicit $\sqrt{N_c}$ in the Hamiltonian enhances the rate of spontaneous emission by the factor N_c. All told, both a simple-minded golden rule calculation of the HO transition rates due to coupling to the photon vacuum and a full-blown analysis of spontaneous emission as in Ref. [15] give the damping rate of the amplitude of the HO as

$$\Gamma = N_c\gamma f(l|\mathbf{k}|)\,. \tag{6}$$

Here $f(x)$ is a function that in the limit $x\to\infty$ behaves as $f(x) \approx \frac{3}{2}x^2$. In equilibrium the expectation number of atoms in the state $|e\psi_0\rangle$ is

$$\langle b_0^\dagger b_0\rangle = N_c\kappa^2/[(\omega-\Omega)^2 + \Gamma^2]\,. \tag{7}$$

At low light intensity, $\langle b_0^\dagger b_0\rangle$ from Eq. (7) is numerically equal to the excitation probability for a two-state atom with the Rabi frequency $\sqrt{N_c}\kappa$ and transition linewidth Γ. The difference between a HO and a two-state atom is that the HO does not saturate with increasing intensity.

Equation (7) has three remarkable implications. First, since $\Gamma\propto N_c$, for a macroscopic Bose condensate the optical resonance is extremely broad. Second, while there are N_c particles in the Bose condensate, on resonance

$(\Omega=\omega)$ the number of atoms directly excited from the condensate is of the order N_c^{-1}. Third, for on-resonance driving light the total number of photons per unit time scattered by the condensate is

$$dN_S/dt=2\Gamma\langle b_0^\dagger b_0\rangle=2\pi\varepsilon_0E^2l^2/\hbar k\,, \tag{8}$$

which is independent of N_c.

The rate of photon scattering should be compared with the number of photons per unit time entering the condensate, dN_I/dt. The latter is estimated by multiplying the flux of incoming photons by the cross section of the condensate, say, πl^2. We find

$$dN_S/dt=4(dN_I/dt)\,. \tag{9}$$

The number of scattered photons seems to exceed the number of incoming photons by a factor of 4. Evidently the thin-sample approximation is not strictly valid. Inasmuch as $l|\mathbf{k}|\gg 1$, the optical thickness of the Bose condensate rather seems to be of the order of 1. Such a mild breakdown of the thin-sample approximation may introduce inaccuracy in our numerical values, but we deem it unlikely that qualitative considerations would be affected.

We are now in a position to discuss the optical signatures of the Bose condensate. The optical thickness of the condensate is approximately unity, so the condensate perturbs the incident light appreciably. However, spontaneous emission predominantly takes place in the forward direction, in a cone with the opening angle $\sim(l|\mathbf{k}|)^{-1}$. This (not coincidentally) also is the order of magnitude of the diffraction angle from an object of size l. All told, the condensate should cast interference fringes with the incident light resembling diffraction from a dielectric object of size l. Fluorescence from the condensate to the sides, and reflection, may take place only as a result of multiple scattering, and should be virtually absent.

Previous calculations [8,9] predict that the condensate reflects near-resonance light. These analyses are based on a partial diagonalization wherein photons and excited atoms are combined into composite degrees of freedom. The dispersion relation of the composite modes has a gap at the atomic resonance frequency, which implies reflection. The difference from our approach is that in the earlier calculations exact momentum conservation follows from the exactly zero momentum of the condensate. Excited atoms with a given momentum $\mathbf{p}$ therefore only couple to photons with momentum $\mathbf{p}$ [16]. Such an atom-photon problem with 2 degrees of freedom for each $\mathbf{p}$ can be diagonalized, and a gap predictably opens up. In contrast, in our model the excited atom is coupled to a range of photons with a smooth distribution of frequencies, a situation that tends to lead to broadening of the excitation rather than a gap.

They key here is the inequality $\Delta\omega=c/l\gg\Gamma$. This inequality validates the Markov approximation in our argument, and ultimately defines the case we call tight confinement. Conversely, as the condensate gets bigger so that conservation of momentum becomes sharper, the results of Refs. [8] and [9] should be regained. The inequality $c/l\gg\Gamma$ may also be read in an alternative manner, namely, that light has time to fly across the condensate during the damping time Γ^{-1}. This is an obvious necessary condition for the entire condensate to act in concert. The limit of tight confinement might equally well be dubbed the limit of complete cooperation.

So far we have mostly ignored the atoms that do not belong to the condensate, as well as transitions between the condensate and the normal-state fraction. A few comments on these topics are due. Transitions within the normal-state fraction produce light scattering that may mask the phenomena associated with the condensate. Such scattering is difficult to analyze quantitatively (cf. [17]), but qualitative arguments may be made. The reason for the suppression of scattering from the condensate is that a large numerical factor $\sqrt{N_c}$ in the matrix element for the transition between the condensate and $|e\psi_0\rangle$ leads to strong spontaneous damping of the transition. For our example the maximum thermal occupation number of a noncondensate state is $k_BT/\hbar\nu\sim 200$, so the analogous suppression of fluorescence should be much less prominent in the noncondensate fraction. As a result, near resonance the normal gas is likely to be optically extremely thick, and simply hides the condensate from view. The best bet for observing Bose condensation probably is to look at large atom-field detunings, of the order Γ, at which the normal-state fraction has become inert.

To estimate the leakage rate of atoms from the state $|e\psi_0\rangle$ to the noncondensate states let us temporarily assume that the rate of such transitions equals the one-atom spontaneous emission rate 2γ. We thus have an estimate for the depletion rate $R\sim 2\gamma\langle b_0^\dagger b_0\rangle$, which for detunings $|\omega-\Omega|\leq\Gamma$ becomes $R\sim 2\gamma[(\kappa/\gamma)^2/N_cf^2]$. This estimate actually is low: The maximum thermal occupation number of a noncondensate state is of the order $k_BT/\hbar\nu$, which translates into enhancements of spontaneous emission rates into the individual noncondensate states by factors of up to $k_BT/\hbar\nu$. Our initial estimate R must be multiplied by a factor that depends on the thermal occupation numbers of all states accessible by spontaneous emission from $|e\psi_0\rangle$. We do not analyze the details here, but only note that the factor obviously is between 1 and $k_BT/\hbar\nu+1$. Nonetheless, the large number N_c in the denominator of the depletion rate R testifies to the robustness of the condensate against optical interrogation.

While our argument was phrased for a condensate bigger than the wavelength, $l|\mathbf{k}|\gg 1$, the opposite limit is even easier to handle. Conservation of momentum is then inconsequential, and the condensate, being smaller than the wavelength, cannot distinguish between directions. On resonance, at low light intensity, a condensate of N_c atoms looks exactly like one atom. At higher intensity the difference emerges that the transition between the condensate and $|e\psi_0\rangle$ does not saturate like the transition in one atom would do. On the other hand, even far away ($\sim\Gamma = N_c \gamma$) from resonance the condensate still scatters photons like one near-resonance atom.

We now briefly consider the interactions between condensate atoms. Wanting an *ab initio* treatment, we propose the following strategy: The hallmark of quantum superfluids is that an order parameter akin to a macroscopic wave function can be associated with the system [18]. For the ideal Bose gas $\psi_0(\mathbf{r}) = \sqrt{N_c}\langle \mathbf{r}|0\rangle$ is a natural choice for such a macroscopic wave function, but an analogous $\psi_I(\mathbf{r})$ should exist for any interacting condensate. In our theory $\psi_0(\mathbf{r})$ has in effect been coupled with the electromagnetic field as a rigid entity; absorption or emission of photons leads to creation or annihilation of momentum-translated replicas of $\psi_0(\mathbf{r})$. We postulate that in an interacting system $\psi_I(\mathbf{r})$ is simply used in lieu of $\psi_0(\mathbf{r})$. The macroscopic wave function $\psi_I(\mathbf{r})$ is to be determined from the theory of an interacting Bose condensate, but this extra step notwithstanding, the results will be qualitatively similar to the results from the theory of an ideal Bose gas. The main difference is that the actual size of the interacting condensate [19,20] is employed as the size parameter l. Resonant dipole-dipole interactions (cf. [21]) also warrant a mention. It can be easily seen that they shift the resonance frequency ω of the excitation mode $|e\psi_0\rangle$ already in first order in nondegenerate time independent perturbation theory, but we do not attempt a quantitative calculation. We have ignored the frequency shift analogous to the Lamb shift associated with the linewidth Γ anyway, so we do not know exactly where the resonance of the condensate is.

In sum, we consider a tightly confined Bose condensate in which the flight time of light across the condensate is the shortest relevant time scale. Such a condensate of N_c atoms scatters much less light than N_c independent atoms, but in exchange the condensate continues to scatter much farther into the wings of the transition than an individual atom. If the condensate is big enough that diffraction from it may exhibit directional preferences, the condensate predominantly scatters light in the forward direction. These properties could be utilized for optical detection of Bose condensation.

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[16] The coupling of atom-field modes in Refs. [8] and [9] is more involved than our discussion implies. Our analysis is based on the standard rotating-wave approximation (RWA) of quantum optics [15]. Hence, we ignore any process in which an atom is excited and a photon is simultaneously emitted, and its Hermitian conjugate. These processes are kept in Refs. [8] and [9], which leads to inconsequential complications. Incidentally, within the RWA neither ω nor Ω alone is a relevant parameter; they enter the solutions only through the detuning $\Omega - \omega$.

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