Optimal Paths and Domain Walls in the Strong Disorder Limit

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An optimization problem that may be cast in the context of domain walls in ferromagnets and spin glasses, lattice animals, and percolation is described. Numerical calculations in two and three dimensions show that a new universality class is obtained. In the strong disorder limit interfaces are not self-affine but fractal. Further, the nontrivial ground state of frustrated spin glasses is straightforwardly obtained in this limit.

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The effect of disorder on critical phenomena has been a subject of much recent interest. In the limit of weak disorder, pinned domain walls in random exchange ferromagnets exhibit criticality, governed by a zero temperature ($T=0$) fixed point, with an algebraic dependence of the roughness on the length scale [1]. The geometry of domain walls in two dimensional weakly disordered ferromagnets, directed polymers in two dimensional random media, and the dynamics of growing interfaces on a one dimensional substrate as governed by the Kardar-Parisi-Zhang equation are all related to each other and the Burgers' equation in one spatial and one temporal dimension [1–3].

Additional complexity is present in spin glasses—random ferromagnetic and antiferromagnetic couplings lead to frustration and a rugged energy landscape [4]. Complex systems, such as spin glasses [5] and proteins [6], and optimization problems, such as the traveling salesman problem [7], and computing structures [8], are believed to have hierarchical structures [9] characterized by ultrametricity [10].

In this Letter, we consider a strong disorder variant of the domain wall problem that has novel ramifications in several contexts including domain walls, fractally rough surfaces, polymers in random media, the ground state of spin glasses, percolation, and lattice animals. Our numerical results in two and three dimensions are suggestive of new nontrivial critical behavior. The pinned domain walls are in a new universality class. Unlike the weak disorder limit, there are significant overhanging configurations. Even though the frustrated spin glass ground state has a zero magnetization unlike the fully magnetized ferromagnetic state, spin glasses and random ferromagnets behave alike in many respects.

There are several, *essentially equivalent,* ways of viewing the domain wall problem in the strong disorder limit. We first describe the standard optimization problem that is entirely equivalent to the Huse-Henley [1] domain wall in two dimensions (2D) at $T=0$. Consider a two dimensional square lattice of size $L\times L$ with bonds of random strength. Let us assume that the strength of a bond corresponds to the time taken to traverse it. What is the optimal path, say from left to right, that minimizes the total travel time? In the limit of zero randomness, i.e., when all bonds have the same strength, the optimal path is a straight line from left to right. In the limit of weak randomness, one may expect that overhanging configurations do not play a key role, i.e., the optimal path does not have a significant number of segments that go "backwards" (from right to left). Indeed, if one assumes that overhangs are not allowed, one may numerically solve this problem using a transfer matrix method and one reproduces the Huse-Henley result that the optimal path is self-affine characterized by a width (the root mean square deviation from its mean position), W, that scales as $L^{2/3}$. This last result has also been obtained analytically by Huse, Henley, and Fisher using a mapping to Burgers' equation [1].

We now turn to the strong disorder limit of this optimization problem. We assume that the bond strengths (or the traversal times) are so widely distributed that the total travel time along a path may simply be evaluated by the value of the largest bond belonging to the path. This definition is rigorously valid when the time space has an ultrametric structure [10]: the minimum time for traveling from A to B (an optimal path is chosen between A and B to minimize the time—further all subpaths between sites on this optimal path are also optimal), $C(A,B)$, satisfies the relation $C(A,B)\leq \mathrm{Max}(C(A,X), C(X,B))$ for any arbitrary X. We find numerically (the details are presented toward the end of this Letter) that the optimal path is no longer self-affine but is a fractal with fractal dimension D_f. The average length of optimal paths, l, spanning a distance L scales as L^{D_f}.

We now present an alternative definition of the optimal path in the context of domain walls in spin systems. Consider an Ising system described by the Hamiltonian $H=\sum_{\langle ij\rangle}J_{ij}\sigma_i\sigma_j$, where $\langle ij\rangle$ denotes nearest neighbor interactions and the exchange constants J_{ij} are randomly positive or negative in a spin glass or all positive in a ferromagnet.

We now make an assumption similar to that made in

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connection with the ultrametric time-space in the optimization problem which enables a simple attack on the problem of spin glasses, yet leads to nontrivial results. The sum of the magnitudes of any chosen set of exchanges will be approximated by that of the largest exchange (in magnitude) in the set. Such an assumption would be valid for a broad distribution of exchange constants. (For very broad but bounded distributions, the physics would be expected to cross over to the weak disorder limit beyond a certain correlation length.)

Following Ambegaokar, Halperin, and Langer [11], we may then obtain the ground state of the spin glass in the following way. First, we rank the exchange constants in order of decreasing absolute strength. We begin with the largest $|J_{ij}|$ and lock the relative orientation of the spins so that the exchange interaction is satisfied. We then proceed with the next highest $|J_{ij}|$ and repeat the procedure. When independent clusters coalesce, we make use of the twofold degeneracy of the clusters (flipping clusters as a whole is permitted) to satisfy the bonds being currently considered. Nevertheless, eventually, frustration sets in—the relative orientation of a pair of spins established previously is found to conflict with the exchange interaction coupling them. We finally reach a stage, before some of the weakest exchange constants are even considered, when the entire configuration of the spins is decided. On an average, in the spin glass phase, one would expect that half of these weak bonds are frustrated.

In 12×12 spin glasses with a Gaussian distribution of exchange couplings, the ground state energy as determined assuming that the exchanges are widely distributed is only 1.8% higher than the ground state energy obtained using transfer matrix methods.

Consider a sample of linear size L in D dimensions with periodic boundary conditions in all directions. A domain wall can be introduced into the system by changing the boundary conditions to antiperiodic in one direction, while retaining periodic boundary conditions in all the other directions. The construction of the ground state with antiperiodic boundary conditions is initially similar to that with periodic boundary conditions but eventually there is an energy mismatch. On completing the two constructions we compare the ground states and determine the location of the domain wall. The average length of the domain wall, l, spanning a lateral distance L scales as before as L^{D_f}. The optimal path in 2D may be seen to be equivalent to the domain wall formed in a disordered ferromagnet at zero temperature with the exchange constants corresponding to the travel times. (The domain wall is in the dual lattice of the spin system.)

Interesting consequences of broad distributions of physical attributes have been explored in a variety of situations including transport in amorphous semiconductors at low temperatures, electrical conduction, and fluid flow in porous rocks and the magnetic properties of doped semiconductors [11]. In fact, the optimal path we have defined is precisely the route of maximum flow of current when there is a broad range of conductances in an electrical network and therefore provides a measure of the experimentally accessible scale-dependent tortuosity of such a strongly disordered medium. While such broad distributions ought to be well defined for systems smaller than an arbitrary though finite correlation length, the new universality class is further evidence of the rich behavior in ultrametric space [12,13] in which our formulations are exactly valid.

The domain wall in the strong disorder limit and the travel-time minimization problem invoked broad distributions or equivalently assumed an ultrametric space. Further these problems involved quenched distributions of exchange constants or travel times. As mentioned previously, the Huse-Henley domain wall problem may be mapped into a dynamical Burgers equation involving annealed noise [1]. Can an analogous mapping into an annealed problem be effected in the strong disorder limit? We show below two distinct ways of doing this related to lattice animals and percolation. We alert the reader that these versions of lattice animals and percolation are different and are not directly related to their conventional counterparts.

We consider spanning lattice animals [14] with loops. In a square of linear size L, periodic boundary conditions are applied in all directions. We denote one of the directions by z. We select bonds randomly with the following rules: (i) a bond that provides a connecting path along the z direction is not allowed, and (ii) the process terminates when only two clusters of selected bonds exist, i.e., each lattice site belongs to one of two clusters. The number of absent bonds that would connect the two clusters provides a measure of the interfacial length. This lattice animal algorithm is operationally identical to the procedure followed in establishing the ground state of the spin glass. The key difference is that we do not allude to broad distributions or ultrametricity here. Further, unlike the spin glass case, where exchange constants were assigned from a broad distribution in quenched manner before determining the interfacial geometry, here the bonds are picked randomly without any preassignment. One might understand the exact equivalence of these approaches by imagining that the bonds were being picked in decreasing order of magnitude.

An alternative view that leads to a novel form of percolation is as follows: Start with all the bonds present on a square lattice. Consider removing each of the bonds one at a time in random order. If a bond that breaks the connectivity from say left to right is selected, an alternate bond is chosen until a single connecting path remains. In ordinary percolation [15], one follows this procedure until the connectivity is just about to be broken for the first time. Here, one proceeds further essentially removing all isolated clusters and dead ends and paring down the percolation backbone into a topologically one dimensional path (whose spectral dimension is 1). This is an entirely

equivalent way of viewing the optimization problem discussed above in terms of eliminating all bonds which have a large cost in time.

Within the quenched formulation of the optimal path problem, travel times were assigned at the outset from a broad distribution. In the percolation approach, bonds are randomly removed one at a time ensuring that the connectivity is not destroyed. The two approaches are clearly identical if one imagines that the bond removal in percolation corresponds to discarding bonds starting with the largest travel time and working downwards in order. Our percolation algorithm differs from that of invasion percolation [16] in that the latter invokes a local choice whereas the former involves a global selection process.

While the percolation and domain wall approaches are equivalent in two dimensions, in higher dimensions they are not. For example, in three dimensions, a domain wall would divide a box into two parts and have fractal dimension ≥ 2, whereas the percolation path would still be topologically one dimensional.

In three dimensions, the percolation problem analogous to the domain wall involves the selection of bonds until a spanning surface made up of plaquettes normal to the bonds divides the system into two disconnected parts. This situation is different from the usual bond percolation in $D=3$, where one merely requires a connected path from one surface to the opposite one. Our numerical calculations indicate that the appropriate threshold is ≈ 1 for large systems in $D=3$.

We have considered an ensemble of systems in $D=2$ and 3 on a hypercubic lattice. In both $D=2$ and 3 we have checked the validity of our results using several different computer programs designed in the context of percolation, lattice animals, and spin systems. (As expected, the results were found to be identically the same in all the contexts.) We find that $D_f=1.2\pm0.02$ in $D=2$ and $D_f=2.5\pm0.05$ in $D=3$ (see Fig. 1).

In $D=2$, L ranges between 4 and 80 and the statistics are based on between 50000 and 5000 realizations. In 3D the largest L studied is 16 and the statistics are similar. (It is interesting to note that regular spin glasses characterized by a Gaussian distribution of exchange couplings have $D_f=1.26\pm0.03$ in $D=2$. This result was obtained [17] for systems with L up to 12.) Numerical studies on a triangular lattice [18] involving site percolation (instead of bond percolation) yield the same value of D_f as in the square lattice underscoring the universality.

The domain wall in strongly disordered systems is a fractal (Fig. 2). This result is qualitatively different from the weak disorder limit where overhangs do not play a role. In 2D for 60×60 samples fewer than 1% of the cases did not have overhangs, even though only 3.5% of the samples of size 4×4 had overhangs. In 3D, for $L=10$, more than 95% of the walls had overhanging configurations.

Note that, even if there is a finite correlation length beyond which the domain wall is no longer fractal but self-affine, the roughness exponent should be approximately equal to 0.8 in $D=2$, a value distinct from 2/3 —the weak disorder exponent [1]. This follows from our result for D_f and the analysis in Ref. [19]. The value of D_f and the construction procedure are the same for ferromagnets and spin glasses. This arises from the fact that spins connected by a frustrated bond in this construction are also connected by a path of stronger bonds which are unfrustrated.

The domain walls in the weak disorder limit do not have statistically significant overhangs and are related to the behavior of directed polymers in a random environment in the strong coupling limit [2]. Our case differs from this class of problems in two essential ways. First, overhangs are allowed and the polymers are undirected. Second, the metric for the cost of a path between two

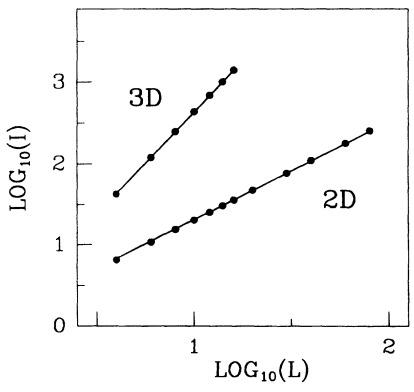

FIG. 1. Log-log plot of I versus L. The results are for 2D and 3D systems. The straight lines have slopes of 1.2 and 2.5, respectively. The statistical errors are smaller than the size of the data points.

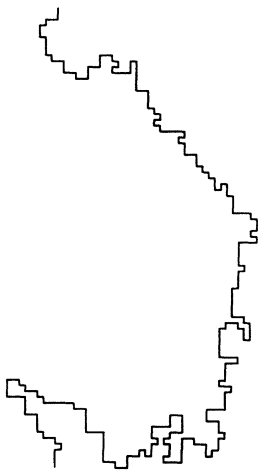

FIG. 2. Typical domain wall in $D=2$ and for $L=80$. Free ends are connected by periodic boundary conditions.

points is not the sum of the costs of the bonds traversed, but the highest cost bond in the path.

It may be tempting to associate the final path of the optimization problem with a self-avoiding walk on a percolation cluster. However, it is straightforward to show that the weighting of different connected paths is not the same for the two cases. Indeed, our numerical estimate of $D_f \approx 1.2 \pm 0.02$ is not consistent with that for a self-avoiding walk on a percolation cluster, for which $D_f \approx 1.30 \pm 0.01$ [20]. Also, while the 3D value of D_f coincides numerically with that of the hull of a percolation cluster [21], our 2D results for both square and triangular lattices suggest that this result is not generally true. Indeed, D_f for the hull of a percolation cluster on a triangular lattice is 7/4, a value quite different from 1.2.

For $D=3$, there has been considerable effort [22] expended in seeking algorithms for finding surfaces in three dimensional space that have a nontrivial fractal dimension greater than 2. While a self-avoiding walk in $D=2$ has a nontrivial fractal dimension, the analogous self-avoiding surface in $D=3$ belongs to the branched polymer universality class [23] with $D_f=2$. Similarly, self-avoiding tethered membranes [22] in $D=3$ have $D_f=2$. The domain wall we have considered is a simple example of a surface with nontrivial geometry in three dimensions.

In summary, we have defined problems in lattice animals, percolation, and domain walls that belong to a new universality class involving fractally rough interfaces, overhanging configurations, and novel exponents. Our simple observation in spin glasses is that when conflicting instructions leading to frustration are widely separated in strength, the system's nontrivial ground state is simply obtained. An extrapolation from this limit to that of regular disorder and relevant frustration seems to be an avenue worth exploring.

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Note added.—After submission of this paper, we received a preprint by C. M. Newman and D. L. Stein in which a similar model of spin glasses was shown to have a multiplicity of ground states above eight dimensions.

[1] D. A. Huse and C. Henley, Phys. Rev. Lett. **54**, 2708 (1985); M. Kardar, Phys. Rev. Lett. **55**, 2923 (1985); D. A. Huse, C. L. Henley, and D. S. Fisher, Phys. Rev. Lett. **55**, 2924 (1985).

[2] M. Kardar and Y. C. Zhang, Phys. Rev. Lett. **58**, 2087 (1987); D. S. Fisher and D. A. Huse, Phys. Rev. B **43**, 10728 (1991).

[3] M. Kardar, G. Parisi, and Y. C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).

[4] A. Bray, Condens. Matter. Phys. **14**, 21 (1988); K. Binder and A. P. Young, Rev. Mod. Phys. **58**, 801 (1986).

[5] M. Mezard, G. Parisi, N. Sourlas, G. Toulouse, and M. Virasoro, Phys. Rev. Lett. **52**, 1156 (1984).

[6] A. Ansari, J. Berendzen, S. F. Bowne, H. Fraunfelder, I. E. T. Iben, T. B. Sauke, E. Shyamsunder, and R. D. Young, Proc. Natl. Acad. Sci. U.S.A. **82**, 5000 (1985).

[7] S. Kirkpatrick and G. Toulouse, J. Phys. (Paris), Lett. **46**, 1277 (1985).

[8] B. A. Huberman and T. Hogg, Phys. Rev. Lett. **52**, 1048 (1984).

[9] R. G. Palmer, D. L. Stein, E. Abrahams, and P. W. Anderson, Phys. Rev. Lett. **53**, 958 (1984).

[10] R. Rammal, G. Toulouse, and M. A. Virasoro, Rev. Mod. Phys. **58**, 765 (1986).

[11] V. Ambegaokar, B. Halperin, and J. Langer, Phys. Rev. B **4**, 2612 (1971); J. Hirsch and J. V. Jose, Phys. Rev. B **22**, 5339 (1980); C. Dasgupta and S. K. Ma, Phys. Rev. B **22**, 1305 (1980); R. N. Bhatt and P. A. Lee, Phys. Rev. Lett. **48**, 344 (1982); D. S. Fisher, Phys. Rev. Lett. **69**, 534 (1992); A. J. Katz and A. H. Thompson, Phys. Rev. B **34**, 8179 (1986); D. Berman, B. G. Orr, H. M. Jaeger, and A. M. Goldman, Phys. Rev. B **33**, 4301 (1986).

[12] B. A. Huberman and M. Kerszberg, J. Phys. A **18**, L331 (1985); A. Maritan and A. Stella, J. Phys. A **19**, L269 (1986).

[13] A. Ogielski and D. L. Stein, Phys. Rev. Lett. **55**, 1634 (1985).

[14] T. Lubensky, in *Ill Condensed Matter*, edited by R. Balian, R. Maynard, and G. Toulouse (North-Holland, Amsterdam, 1979).

[15] D. Stauffer, *Introduction to Percolation Theory* (Taylor and Francis, Philadelphia, 1985).

[16] R. Chandler, J. Koplik, K. Lerman, and J. F. Willemsen, J. Fluid Mech. **119**, 249 (1982).

[17] A. J. Bray and M. A. Moore, Phys. Rev. Lett. **58**, 57 (1987).

[18] A. Bhattacharya (unpublished).

[19] P. Z. Wong and A. Bray, Phys. Rev. Lett. **59**, 1057 (1987); T. W. Burkhardt, Phys. Rev. Lett. **59**, 1058 (1987).

[20] M. D. Rintoul and H. Nakanishi (unpublished). We are grateful to H. Nakanishi for providing us with this estimate.

[21] P. N. Stenski, R. M. Bradley, and J. M. Debierre, Phys. Rev. Lett. **66**, 1330 (1991).

[22] *Statistical Mechanics of Membranes and Surfaces*, edited by D. R. Nelson, T. Piran, and S. Weinberg (World Scientific, Singapore, 1989).

[23] J. O'Connell, F. Sullivan, D. Libes, E. Orlandini, M. Tesi, A. Stella, and T. Einstein, J. Phys. A **24**, 4619 (1991); A. Stella, E. Orlandini, J. Beichl, F. Sullivan, M. Tesi, and T. L. Einstein, Phys. Rev. Lett. **69**, 3650 (1992).