Spin-Glass Model with Dimension-Dependent Ground State Multiplicity

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We introduce a new Ising spin-glass model and use it to study the relation between quenched disorder, frustration, ground state multiplicity, and dimension. The Hamiltonian is of Edwards-Anderson type in any finite volume but the coupling magnitudes scale nonlinearly with volume. We find a mapping between the ground state structure of this model and the global connectivity structure of invasion percolation in the same dimension. We argue that our model has a single pair of ground states below eight dimensions and infinitely many above.

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Disorder and frustration, ingredients central to any spin-glass model, play an important but poorly understood role in determining ground state structure [1]. Early investigations demonstrated that the Sherrington-Kirkpatrick [2] infinite-ranged Ising spin glass displays many thermodynamic states at low temperature [3]. A natural speculation was then that more realistic short-ranged models [4] in finite dimension d exhibit a similar ground state structure, particularly in the (physically relevant) dimension $d=3$. Both numerical [5,6] and high-dimensional analytical [7] studies argued in support of this claim. However, proponents of a scaling ansatz [8–10] concluded that Edwards-Anderson (EA) Ising spin glasses possess only a single pair of (spin-inversion-related) ground states [11] in *all* finite dimensions [12], and that the $d \rightarrow \infty$ limit in these models is singular. Because of the difficulty of any direct analysis of the EA Hamiltonian, the question of its ground state multiplicity in finite d remains one of the central unresolved issues concerning the nature of the spin-glass state.

In this paper we introduce a new spin-glass model which is designed to clarify the relation between disorder and frustration on the one hand and multiplicity of ground states on the other. As such, the model was chosen for its mathematical tractability rather than for physical realism. It demonstrates that the joint presence of disorder and frustration is not sufficient to draw any *a priori* conclusions about ground state structure and/or multiplicity. In particular, we will show that the ground state behavior of our model has a *transition* at eight dimensions: In lower dimensions it possesses only a single pair of ground states in the thermodynamic limit, while in higher dimensions it has uncountably many. The model is appealing because, while it is complex enough to have a rich behavior in its ground state structure, it is simple enough to study analytically, and moreover, the mechanism by which the number of ground states changes can be precisely identified. Equally important, and the basis of this identification, is a mapping to invasion percolation [13–15].

The model.— We work on a cubic lattice in d dimensions, i.e., of sites $\mathbf{x} \in Z^d$ and edges connecting nearest-neighbor sites only. We provide two descriptions of the model— a Hamiltonian and an algorithmic construction of its ground states. We begin with the latter.

Associate each site with an Ising spin variable $\sigma_{\mathbf{x}} = \pm 1$. On the edges there are nearest-neighbor couplings whose values are determined by independent random variables: $\epsilon_{\mathbf{xy}} = \pm 1$ with equal probability, and $K_{\mathbf{xy}}$ chosen from some continuous distribution (e.g., uniformly distributed between 0 and 1). The $\epsilon_{\mathbf{xy}}$'s give the signs of the couplings and the $K_{\mathbf{xy}}$'s the ordering of their magnitudes: An edge with a smaller $K_{\mathbf{xy}}$ value has a larger magnitude coupling, and will be said to have a higher order.

Given a cube Λ of volume L^d centered on the origin and some boundary condition on $\partial\Lambda$, construct a ground state (modulo a spin flip if the boundary condition is spin flip symmetric) by the following procedure: Find the smallest $K_{\mathbf{xy}}$, and choose the spins $\sigma_{\mathbf{x}}$ and $\sigma_{\mathbf{y}}$ such that $\sigma_{\mathbf{x}}\epsilon_{\mathbf{xy}}\sigma_{\mathbf{y}} > 0$; i.e., the coupling is satisfied. Then find the coupling with the next larger $K_{\mathbf{xy}}$ value, and choose the spins on its end points to satisfy it. Repeat this procedure, *unless* a resulting closed loop (or path connecting two boundary sites) with previously satisfied couplings forbids it. When that happens, simply proceed to the coupling next in order, and continue until every coupling has been tested.

Note that until an edge-connected cluster of spins reaches the boundary, only the spins' relative orientations are known. If the boundary condition breaks spin-flip symmetry, the sign of each spin in a cluster will be determined as soon as it connects to the boundary.

For a given L, a Hamiltonian which has a ground state structure obtainable via this procedure is the usual $\mathcal{H}_L = -\sum_{\langle\mathbf{xy}\rangle} J_{\mathbf{xy}}^{(L)}\sigma_{\mathbf{x}}\sigma_{\mathbf{y}}$, but with the magnitude of each

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coupling constrained to be greater than the absolute sum of all the smaller magnitudes; e.g., it suffices if each $|J_{\mathbf{xy}}^{(L)}|$ is at least twice as large as the next smaller one. It is clearly impossible for such a constraint to be generally satisfied if the $J_{\mathbf{xy}}^{(L)}$'s are independent random variables with a common distribution not depending on L (the usual situation in an EA spin glass). However, it can be satisfied, for example, via an L-dependent nonlinear scaling factor, $\lambda^{(L)}$, by defining

$$J_{\mathbf{xy}}^{(L)}=c_L\epsilon_{\mathbf{xy}}e^{-\lambda^{(L)}K_{\mathbf{xy}}}, \tag{1}$$

where $\epsilon_{\mathbf{xy}}$ and $K_{\mathbf{xy}}$ have the same meaning as before, and c_L is a linear scaling factor which plays no role in ground state selection. If $\lambda^{(L)}$ is chosen to increase sufficiently rapidly as $L\to\infty$, then the above constraint is satisfied for all large L (with probability 1). We note that this model is intended to elucidate properties of ground state structure only, and has no interesting behavior at nonzero temperatures.

Statement of the problem.— The question of whether this model has multiple ground state pairs is equivalent to whether a change in boundary conditions can change a coupling deep in the interior from being satisfied to unsatisfied, or vice versa.

We therefore ask whether any coupling $J_{\mathbf{x}_1\mathbf{x}_2}$ exists with the following property: Before any path of satisfied couplings joining $\mathbf{x}_1$ and $\mathbf{x}_2$ within Λ is formed according to the algorithm described above, there exist two disjoint paths, one joining $\mathbf{x}_1$ to the boundary, and the other joining $\mathbf{x}_2$ to the boundary. If such a coupling exists, then whether it is satisfied or unsatisfied will be determined by the boundary conditions.

When the boundary is sufficiently far from some interior region, it may be that no such coupling exists within the region—each coupling is either itself tested before two such disjoint paths can be found, or else its end points are first connected via some internal path of previously tested couplings. If that is the case, only a single pair of spin-flip-related ground states exists in the thermodynamic limit. Otherwise, the system possesses multiple pairs of ground states.

Mapping to invasion percolation.— We now reformulate our model as follows. Starting from some site $\mathbf{x}$, we construct a growing tree in the following way: From those edges touching $\mathbf{x}$, find the one whose coupling has the highest order and satisfy it. Now consider all edges from that edge connecting to new sites; pick the highest order of those and satisfy it. Proceeding inductively, we generate trees $T_n(\mathbf{x})$ of satisfied couplings containing n edges and $n+1$ sites. This procedure is identical to that employed in invasion percolation (except we include only edges which connect to new sites) on a d-dimensional cubic lattice [13–15].

For a given L and fixed boundary condition assigning spin values to $\partial\Lambda$, $\sigma_{\mathbf{x}}$ is determined by the tree $T_N(\mathbf{x})$ at the (random) time N at which $T_N(\mathbf{x})$ just reaches $\partial\Lambda$ and by the value of the boundary spin it touches at that time. It is not hard to show that the value of $\sigma_{\mathbf{x}}$ obtained in this way will agree with that given by our earlier algorithm. It follows that the question of many (pairs of) ground states in the infinite volume limit is precisely equivalent to that of whether disjoint trees $T_\infty(\mathbf{x})$ can be found for many sites $\mathbf{x}$.

We can then already answer the question of multiplicity of states of this model in two dimensions. Because it is rigorously known that for $d=2$ invasion percolation the trees $T_\infty(\mathbf{x})$ and $T_\infty(\mathbf{y})$ always intersect (in fact are the same modulo finitely many sites) [16], it follows that our model has only a pair of ground states in two dimensions. Whether any two such trees in higher dimensions must intersect is an interesting problem in invasion percolation, which we now consider.

Multiplicity of ground states.— The question of multiplicity of ground states in our model has been mapped to that of whether invasion percolation has nonintersecting invasion regions. Invasion percolation has the feature that the invaded region asymptotically approaches the so-called incipient infinite cluster (i.e., at the critical percolation probability p_c) in the independent bond percolation problem on the same lattice [17]. The fractal dimension D of the incipient cluster in the independent bond problem on the d-dimensional cubic lattice is known from both numerical studies and scaling arguments; in particular, D is dimension dependent below six dimensions, but $D=4$ for $d\geq 6$ [17].

The following heuristic argument might then provide an intuitive picture of our model's behavior. Consider two invasion regions $T_\infty(\mathbf{x})$ and $T_\infty(\mathbf{y})$ emanating separately from points $\mathbf{x}$ and $\mathbf{y}$, both in Z^d, which are very far apart. If the fractal dimension of each region is less than $d/2$, the randomly growing trees will "miss" each other with high probability; otherwise, they will connect up. It would follow that if the fractal dimension of the invasion region is equal to D, then the critical dimension of our model is eight. Below eight dimensions invasion regions should always intersect, and hence there would be only one pair of ground states in our spin-glass model; above that there should be an infinite number. Similar heuristic reasoning suggests that for $d>8$, the number of distinct configurations on the L^d cube coming from (infinite volume) ground states should be of order $2^{L^{d-4}}$.

A potential problem with the above argument is that D is only a *lower bound* for the fractal dimension D_i of the invasion region. Our main task is then to determine D_i as a function of space dimension d. In order to make our argument clearer, however, we first make more precise the intuitive notion that nonintersection should occur if $D_i<d/2$.

Nonintersection in invasion percolation.— Consider invasion percolation as described above, and define *a pair connectedness function* $G(\mathbf{y}-\mathbf{x})$ as the probability that the site $\mathbf{y}\in T_\infty(\mathbf{x})$.

We now make the following claim: If

$$\sum_{\mathbf{x} \in Z^d} G(\mathbf{x})^2 < \infty , \qquad (2)$$

then there exist infinitely many disjoint invaded regions; i.e. (with probability 1), there are sites $\mathbf{x}_1, \mathbf{x}_2, \ldots$ so that $T_\infty(\mathbf{x}_i) \cap T_\infty(\mathbf{x}_j) = \emptyset$.

Because of space limitations, we merely sketch the (rigorous) proof of the above claim. First, one shows that square summability of $G(\mathbf{x})$ implies that for $\mathbf{x}_i$ and $\mathbf{x}_j$ far apart, the probability is small that $T_\infty(\mathbf{x}_i)$ meets (or comes close to meeting) an *independent* [18] region $T'_\infty(\mathbf{x}_j)$; this is done by using Fourier transforms to bound the expected number of (near) meeting sites. Then one notices that the probability *not* to come within distance one of meeting $T_\infty(\mathbf{x}_i)$ is the same for $T'_\infty(\mathbf{x}_j)$ as for $T_\infty(\mathbf{x}_j)$. This follows by an appropriate construction [16] of the invaded regions which shows $T_\infty(\mathbf{x}_j)$ remains independent of $T_\infty(\mathbf{x}_i)$ as long as they do not come within a single edge of one another.

Now define D_i by the relation [17,19]

$$G(\mathbf{y}-\mathbf{x}) \sim 1/|\mathbf{y}-\mathbf{x}|^{d-D_i} \qquad (3)$$

as $|\mathbf{y}-\mathbf{x}| \to \infty$; then the claim that nonintersection of infinite invasion regions must follow when $D_i < d/2$ has been made precise [20]. It remains to determine D_i as a function of d. This we will do nonrigorously by utilizing results already in the literature.

Monte Carlo simulations of invasion percolation on square and simple cubic lattices [15] provide strong evidence that $D_i = D$ in dimensions two and three ($D = 91/48$ and $D = 2.53$) [17]. In higher dimensions, less is known. There does exist, however, an exact solution for invasion percolation on a Cayley tree [21]. From this solution, one can deduce the fractal dimension D_i of the invasion region using two different measures. The first is to let the invasion proceed for n steps, and then compute the radius of gyration R. For the Cayley tree, it was found [21] that, as $n \to \infty$, $R \sim n^{1/4}$, consistent with $D_i = 4$. The second, which is much closer to Eq. (3), is to evaluate the shape function, S_m^n, which is the mean number of invaded sites on level m of the Cayley tree for an invasion of n steps. By analyzing Eq. (9) in Ref. [21] (valid for the simplest Cayley tree) in the limits $\beta \to 1$ and $\alpha \to 1$ (in that order), we find that S_m^∞ is proportional to m (with logarithmic corrections) as $m \to \infty$. The *total* number of sites invaded up to level m thus scales as m^2. The usual measure of distance on a Cayley tree places level m at distance $\sqrt{m}$ from the origin, leading again to $D_i = 4$.

We therefore conclude that $D_i = 4$ is an *upper bound* for the fractal dimension of an invasion tree on a lattice in finite dimension. Given that $D_i \geq D$ in any dimension, and given the known values of D, we conclude that the critical dimension in our problem is eight.

Boundary conditions and chaotic size dependence.— In an earlier paper [22], the authors argued that multiplicity of ground states in the EA Ising spin glass was associated with nonexistence of a single limiting Gibbs distribution, in the thermodynamic limit, for coupling-independent boundary conditions. We find that the same association holds here. While the conclusion is fairly clear for any sequence of *fixed* boundary conditions, in which each boundary spin is assigned a definite value, the mechanism for spin-symmetric (e.g., free or periodic) boundary conditions is more subtle. Its investigation provides deeper insight into the nature of our spin-glass model, and in particular the role played by frustration.

In all dimensions the ground state structure is determined by the subset of couplings which are satisfied in our algorithmic construction *regardless* of their sign. Unlike the set of *all* satisfied couplings, this subset can form no closed loops and hence must yield one or more trees. It is easily seen that each site belongs to one of these trees, and that each tree connects to the boundary. The set of such trees, in the thermodynamic limit, equals the union over all $\mathbf{x}$ of $T_\infty(\mathbf{x})$. We will call this set the "invasion forest" [23]. Below eight dimensions there is only one tree in this forest, while above eight dimensions there are infinitely many disjoint trees. The "state" of each tree can be one of a pair of spin configurations related via a global flip. The boundary spins, if fixed, will determine the sign of each spin on a given tree. If the system passes through a sequence of volumes with fixed, coupling-independent boundary conditions, the "sign" of each tree will randomly flip, and no single limiting ground state is obtained in the thermodynamic limit. Indeed the size dependence is sufficiently chaotic that all of the uncountably many ground states will appear as limits along coupling-dependent subsequences of volumes.

Consider now the case of a finite cube $\Lambda \subset Z^d$, with $d > 8$, and with spin-symmetric (e.g., free or periodic) boundary conditions. At first it might seem that our earlier conclusions are incorrect, because for *every finite volume there is only one tree.* This is because, unlike when the boundary spins are fixed, the procedure of satisfying couplings of increasingly smaller order will be continued until all sites are connected. Nevertheless, there again is no single limiting pair of ground states for a sequence of volumes (chosen independently of the couplings) with spin-symmetric boundary conditions. The reason is that, as $L \to \infty$, the path of edges connecting two trees (which would not be connected with fixed boundary conditions) will move out to infinity [24], so that the *relative* sign between trees flips randomly.

This surprisingly subtle phenomenon highlights the importance of frustration in our model. If one constructed a model of a random *ferromagnet* using the same algorithm but with all $\epsilon_{\mathbf{xy}} = +1$, then one could again generate many ground states (in $d > 8$) with appropriate choices of fixed boundary conditions (just as one can with uniform ferromagnets in any dimension or ordinary random ferromagnets [25] in $d > 5$). However, free or periodic boundary conditions *would* yield a single pair of ground states as $L \to \infty$, namely, all spins up and all spins down. As with the usual models of ferromagnets and spin glasses, the difference in ground state structure is revealed most sharply through the use of spin-symmetric boundary conditions.

Conclusion.— We have introduced a nearest-neighbor Ising model containing both quenched disorder and frustration. While its Hamiltonian has the same form as that of the EA spin glass, the unusual scaling of the couplings allows for a direct mapping onto invasion percolation. Our conclusions apply to both invasion percolation and to our spin-glass model:

(1) The invasion forest on the infinite lattice Z^d has only *one* tree in dimensions less than eight, and *infinitely many* above eight.

(2) In the spin-glass model introduced in this paper, there exists only a *single* pair of (spin-inversion-related) ground states in dimensions less than eight, and *uncountably many* above eight.

In both cases, eight is a marginal dimension; the elucidation of its ground state structure requires further investigation.

We reemphasize that our spin-glass model is highly simplified, and should not be used to draw any firm conclusions about the ground state structure of realistic models, particularly the EA Ising spin glass. We do claim that it demonstrates that quenched disorder and frustration of even the simplest variety can lead to a rich behavior, and that their presence does not alone ensure any preordained conclusions about number or structure of ground states.

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Note added in proof.— After this paper was submitted, we learned that Cieplak, Maritan, and Banavar [26] have independently studied this model. We thank Jay Banavar for bringing this work to our attention.

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