Experiments on Not "Hearing the Shape" of Drums

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We present an experimental test of the theorem of isospectral domains which states that certain pairs of planar domains have identical spectra for eigenvalues of the Laplacian operator or the Helmholtz wave equation. The experiments employ thin microwave cavities shaped in the form of two different domains known to be isospectral. We verify the equality of at least 54 of the measured low-lying eigenvalues to a few parts in 10^4. We obtain the eigenfunctions experimentally, and confirm that two nonisometric transformations connect isospectral eigenfunction pairs. An analysis in terms of periodic orbits using the trace formula is carried out, and shown to be an alternative confirmation of isospectrality.

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The properties of the Helmholtz wave equation, $(\nabla^2+k^2)\psi=0$, are of fundamental importance in quantum mechanics, electromagnetics, and acoustics. In closed two dimensional (2D) domains, the topology, particularly the shape of the boundary, has been recognized to play an important role in determining the features of the spectrum [1]. The area and the perimeter have long been known to determine the mean or smooth behavior of the energy level density. More recently, in the context of quantum chaos [2–4], statistical fluctuations of the energy levels about the mean have been shown to be determined by the classical dynamics of a particle in the domain, leading to universality classes which are different according to whether the dynamics is integrable or chaotic. In general, most such results are concerned with generic properties of spectra and their relationships to classes of domains. In this context, a recent theorem by Gordon, Webb, and Wolpert [5] is particularly remarkable. It establishes that two different 14-sided shapes of equal area have identical spectra for all values of the energy. This answers in the negative the well-known question, "Can you hear the shape of a drum?" posed by Kac [1], since it leads to the remarkable conclusion that two drums built of such shapes would "sound exactly alike" [6].

Although the isospectrality of several planar geometries (besides the original in Ref. [5]) has now been proven on mathematical grounds, the actual eigenvalues composing the spectrum of any isospectral pair, and the associated eigenfunctions, are not known. Not only are analytic solutions unattainable, but the special features of the geometries, particularly the presence of sharp corners, also appear to lead to difficulties in numerical solutions. The present work [7] employs a novel approach, viz., the use of actual physical experiments using microwave cavities, to determine the eigenvalue spectrum and eigenfunctions of the special geometries, thus enabling tests of their isospectrality and of proposed relations between the eigenfunctions.

Mathematical results such as those of Ref. [5] are of importance in physics because they reveal new aspects of the wave equation. The isospectrality of different domains implies an underlying symmetry, and this is confirmed by the presence of a transformation connecting eigenfunction pairs corresponding to each eigenvalue. The proof of isospectrality utilizes the notion of "transplantation" of eigenfunctions from one domain to the other [8,9]. This nonisometric transformation is a recent result in mathematics. By direct measurement of the eigenfunctions for the first time, we are able to test and confirm this transformation, which can only be done in physical experiments such as ours.

Electromagnetic waves of wavelength λ in metallic enclosures (cavities) obey the Helmholtz equation when the cavity has one dimension $d\ll\lambda$. Then time-independent solutions are allowed which are two dimensional, with the electric field E_z playing the role of ψ in the wave equation, with Dirichlet conditions $E_z=0$ on the boundary. The corresponding states are called transverse magnetic (TM), and are the only states observed for all frequencies $f_{\max}<c/2d$. Acoustical drum-head vibrations also obey the same wave equation. The correspondence of the Helmholtz wave equation and the (particle) Schrödinger's equation has been previously exploited [10–12] to study a variety of problems in quantum chaos using microwave cavities. Here we use the electromagnetic system to verify the isospectral theorem and to obtain quantitative results for the eigenvalues and eigenfunctions.

The two cavities were fabricated from copper, with 8-sided cross-sectional shapes communicated to us by DeTurck [13]. These two domains are among a family of pairs of domains in 2D which are recognized to be isospectral [14,15]. The cavities had unit length 3 in. (cavity area 31.5 in.^{2}) and thickness $d=0.25$ in. (nearly 6 mm) leading to $f_{\max}\sim 25$ GHz. It is estimated that dimensional tolerances were about 10^{-2} in. The transmission spectra of the cavities were studied with an HP8510 Network Analyzer. Details of the measurement method are discussed in Ref. [16]. Eigenvalues are obtained by determining the maxima of resonances in the transmission spectra. Multiple coupling positions for the input and output probes were used to ensure that eigenstates were not missed due to accidental location of any pair of

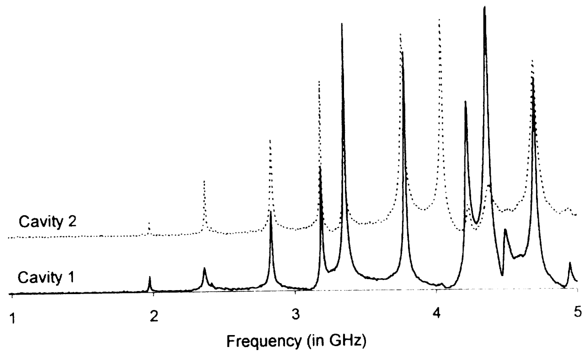

FIG. 1. Experimental transmission spectra vs frequency for the two cavities. The traces are vertically shifted for clarity. The isospectrality is evident from the exact coincidence in frequency of the resonance maxima.

coupling probes at nodes of eigenstates.

The transmission amplitude spectra up to 5 GHz of the two cavities are shown in Fig. 1. Resonances are clearly seen whose maxima are located at the eigenfrequencies. These resonances are broadened due to absorption, and also possibly, the presence of sharp corners. The amplitudes of the resonances are irrelevant [6] for the subsequent discussion, since they are determined only by the field strengths of the corresponding wave function at the two coupling probe locations.

The most striking feature of Fig. 1 is the simultaneous occurrence of resonances at identical frequencies. Thus Fig. 1 constitutes a visual verification of the theorem of isospectrality. More precise measurements of the lowest 54 eigenvalues were carried out via accurate measurements of the resonance maxima on expanded ($\times 100$) frequency scales. This was repeated for several coupling probe locations to ensure that there were no missed levels.

The results of the measurements for the 25 lowest eigenvalues f_n are shown in Table I, and are typical of results up to $n=54$. Also shown are energy eigenvalues of the spectrum, $E_n=f_n^2$, normalized to the ground state value. Differences df_n in the eigenvalues between the two cavities were found to be $\sim 0.01\%$ to 0.2% of f_n. The differences are also small compared to the nearest neighbor spacing of levels ($\sim 3\%$), compared to the widths of the resonances ($<0.1\%$). We believe that the principal reason for observed discrepancies are slight imperfections introduced during assembly of the pieces. The individual pieces were themselves machined to accuracies better than about 0.01%, but assembly is probably restricted to this accuracy. We have separately verified that the perturbations due to the coupling holes are not responsible for these discrepancies.

The results for the 54 eigenvalues were found to be in excellent agreement with the Weyl formula for the in-

TABLE I. Table of lowest 25 eigenfrequencies f_n of the two cavities. Also shown is the normalized energy spectrum $E_n=f_n^2$.

Eigen No.	Cavity 1, f (GHz)	Cavity 2, f (GHz)	Difference (%)	Normalized energy
1	1.99070	1.99080	0.01	1.000
2	2.38350	2.38589	0.10	1.435
3	2.84123	2.84186	0.02	2.037
4	3.19575	3.19518	0.02	2.577
5	3.36790	3.36590	0.06	2.860
6	3.79639	3.79240	0.11	3.633
7	4.07030	4.07028	0.00	4.180
8	4.24985	4.24938	0.01	4.557
9	4.39223	4.39088	0.03	4.866
10	4.51840	4.51570	0.06	5.148
11	4.73100	4.72943	0.03	5.646
12	4.98030	4.98970	0.19	6.270
13	5.14655	5.14590	0.01	6.683
14	5.25935	5.26230	0.06	6.984
15	5.44910	5.45910	0.18	7.506
16	5.71413	5.71975	0.10	8.247
17	5.75988	5.77450	0.25	8.393
18	5.89800	5.89050	0.13	8.766
19	6.09413	6.09175	0.04	9.367
20	6.19568	6.19913	0.06	9.691
21	6.21981	6.23100	0.18	9.779
22	6.38820	6.38975	0.02	10.300
23	6.55065	6.55200	0.02	10.830
24	6.64530	6.64150	0.06	11.136
25	6.81053	6.80675	0.06	11.697

tegrated staircase density of states $N(E)=(A/4\pi)E-(L/4\pi)E^{1/2}+K$ with measured values for the area $A=31.5$ in.^{2}, $L=30.72$ in., and $K=0.54$, where we have included the correction for the perimeter (L) and the topology (K). The agreement suggests that levels were not missed, and also that there were no degeneracies or multiplicities, at least for these low energy levels. Number-theoretic degeneracies are expected at high energies because a class of eigenvalues for these geometries are also eigenvalues of the unit (45°,45°,90°) triangle. But still the absence of degeneracies at the lower energies comes as a surprise, since, as discussed later, there exist two transformations which connect eigenfunctions in each domain.

A particularly powerful capability of the present experiments is the ability to measure wave functions in arbitrary geometries. This is achieved using a cavity perturbation technique first introduced by one of us in Ref. [10], which led to the direct experimental observation of scars in wave functions of chaotic geometries [10]. The results for the $n=1$, 3, and 6 eigenstates for both cavities are displayed in Fig. 2. Note that even though the corresponding frequencies agree to 10^{-4} the spatial details of the eigenfunctions in the two cavities are completely different.

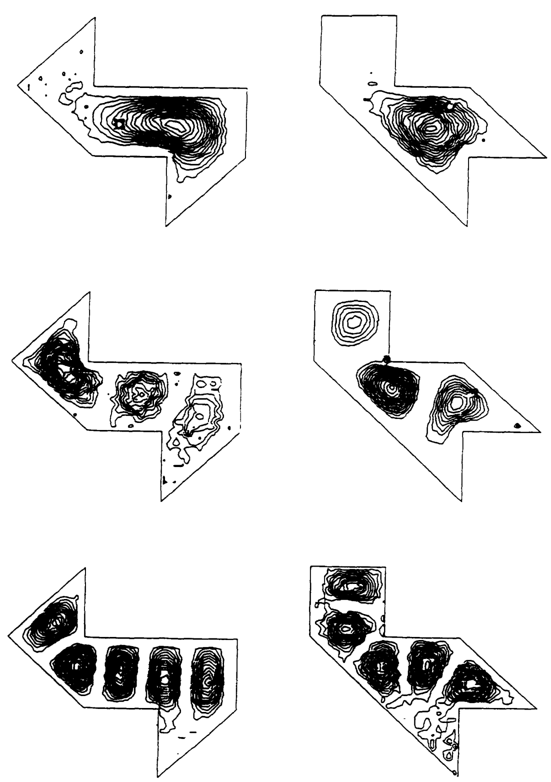

FIG. 2. Pairs of experimental wave functions for selected eigenvalues of the two cavities. The eigenfrequencies in GHz are 1.9907, 1.9908 (top), 2.8413, 2.8418 (middle), 3.79639, 3.79240 (bottom).

The theorem of isospectral domains is particularly remarkable because it states that eigenfunctions which are not isometrically related may nevertheless possess the same eigenvalues. Beyond the (trivial) observation that the number of maxima is identical for isospectral eigenstates, it is clear that a simple isometric (translation +rotation) transformation does not connect such eigenfunctions. However, there does exist a nonisometric transformation [13], based upon the observation that each geometry can be divided into seven triangles. "Transplantation" operations are carried out on the triangular "pieces" of the wave function in one geometry. In addition to rotations and translations, an additional (symmetry) operation is an inversion about the center line through the 90° vertex, denoted by $\phi\rightarrow\tilde{\phi}$. Then each triangular piece of the wave function of cavity 2 can be constructed from a linear combination of transformations of pieces of cavity 1 (see Fig. 3). There are two such transformations [17], one a three-element transformation and the other a four-element transformation. The three-element transformation can be obtained by substituting $a=0$ and $b=1$, and the four-element by substituting $a=1$ and $b=0$, in Eq. (1):

$$\begin{bmatrix}\psi_{A2}\\ \psi_{B2}\\ \psi_{C2}\\ \psi_{D2}\\ \psi_{E2}\\ \psi_{F2}\\ \psi_{G2}\end{bmatrix}=\begin{bmatrix} a & -\tilde{a} & a & -\tilde{b} & b & -\tilde{a} & -\tilde{b}\\ -\tilde{a} & a & -\tilde{b} & a & -\tilde{a} & b & b\\ a & -\tilde{b} & a & -\tilde{a} & b & -\tilde{b} & -\tilde{a}\\ -\tilde{b} & a & -\tilde{a} & b & -\tilde{a} & b & a\\ b & -\tilde{a} & b & -\tilde{a} & b & -\tilde{a} & -\tilde{a}\\ -\tilde{b} & b & -\tilde{a} & a & -\tilde{a} & a & b\\ -\tilde{a} & b & -\tilde{b} & b & -\tilde{a} & a & a\end{bmatrix}\begin{bmatrix}\phi_{A1}\\ \phi_{B1}\\ \phi_{C1}\\ \phi_{D1}\\ \phi_{E1}\\ \phi_{F1}\\ \phi_{G1}\end{bmatrix}. \tag{1}$$

Here a minus sign is accompanied by a flip about the symmetry axis of the triangle, which is represented by the tilde.

Utilizing the three-element transformation, we have constructed the ground state eigenfunction of cavity 2 using the measured wave function in Fig. 2 for the ground state of 1. The constructed ground state shown in Fig. 3 is in good agreement with the corresponding measured ground state of 2 (compare Fig. 2). The result of the four-element transformation is found to be identical, except for a reversal of sign. This is consistent with the experimental finding of no degeneracies in the spectral window of the present experiment.

It is interesting to note that the Gutzwiller trace formula, a result well known in the context of quantum chaos, also suggests that the two domains are isospectral. Because of the presence of the 270° and the 135° angles in the two domains, the two geometries are pseudo-integrable [18]. Hence the invariant integral is not a torus, but a surface of genus 7 in both cases. The period-

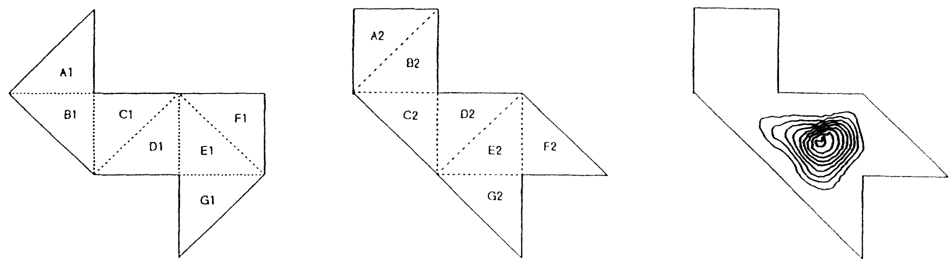

FIG. 3. Constructed ground state of cavity 2 using the measured ground state of cavity 1, and using the nonisometric transformation on wave function pieces in triangular sections of cavity 1.

ic orbits in these geometries form one parameter families, covering bands on the phase space surface which are bounded by trajectories which hit a vertex. In this case the trace formula can be written as [19] $\sum\delta(E-E_n)=1+\sum_{j=1}^{\infty}a_jJ_0(\sqrt{E}l_j)$, where the l_j and a_j are the lengths and phase-space areas (normalized) of the periodic orbits [18]. A numerical study of the periodic orbits was carried out and shows that at least a hundred of the l_j,a_j are identical for these geometries, and suggests that another check of isospectrality (accurate to order $E^{-1/4}$) can therefore be achieved by an analysis of the periodic orbits. An interesting point to be noted is that the number of bounces, in corresponding periodic orbits of the two cavities, is not always the same, even though the length and the phase-space areas are identical. This represents a novel and unexpected application of the trace formula, which is widely used in quantum mechanical problems.

The present work goes beyond a mere demonstration of a mathematical theorem. The experiments yield the actual spectrum, which can be used to address the issue of whether the eigenvalue spectra possess special mathematical attributes besides their isospectrality. The experiments have also illuminated the issue of degeneracies. The experimental determination of the actual eigenfunctions is essential to test the eigenfunction transformation. None of these issues appear to be easily amenable to numerical solutions. This interplay of mathematics and physics is beneficial to both fields. While the experiments have provided a satisfying physical basis for the mathematical results, the new ideas from mathematics which have been studied here may have wide and unforeseen impact on physical problems [15].

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