One-Loop n-Gluon Amplitudes with Maximal Helicity Violation via Collinear Limits

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We present a conjecture for the n-gluon one-loop amplitudes with maximal helicity violation. The conjecture emerges from the powerful requirement that the amplitudes have the correct behavior in the collinear limits of external momenta. One implication is that the corresponding amplitudes where three or more gluon legs are replaced by photons vanish for $n > 4$.

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Multijet processes at colliders require knowledge of matrix elements with multiple final state partons. At tree level concise formulas for maximally helicity violating amplitudes with an arbitrary number of external legs were first conjectured by Parke and Taylor [1], and later proven by Berends and Giele using recursion relations [2,3].

In general, amplitudes in gauge theories satisfy strong consistency conditions; they must be unitary, and must satisfy correct limits as the momenta of external legs become collinear [1,2,4]. In this Letter we discuss the example of a one-loop amplitude which is sufficiently constrained that we can write down a form for an arbitrary number of external legs. The all-n conjecture which we present is for maximal helicity violation, that is with all (outgoing) legs of identical helicity, was originally displayed in Ref. [5], and has just been confirmed by recursive techniques [6,7]. The construction is based upon extending the known one-loop four- and five-gluon [8] amplitudes which were first obtained using string-based methods [9].

The one-loop n-gluon partial amplitude $A_{n;1}(1^+, 2^+, \ldots, n^+)$ is associated with the color factor $N\mathrm{Tr}(T^{a_1}\cdots T^{a_n})$ and gives the leading contribution to the amplitude for large N [4,10,11]. The subleading partial amplitudes $A_{n;c}$, $c > 1$, can be obtained from $A_{n;1}$ by summing over various permutations [11,12]. The structure of $A_{n;1}$ is particularly simple, making it an ideal candidate for finding an all-n expression. The all-plus helicity structure is cyclicly symmetric; and no logarithms or other functions containing branch cuts can appear. This can be seen by considering the cutting rules: The cut in a given channel is given by a phase space integral of the product of the two tree amplitudes obtained from cutting. One of these tree amplitudes will vanish for all assignments of helicities on the cut internal legs since $A_n^{\text{tree}}(1^\pm, 2^+, 3^+, \ldots, n^+) = 0$, so that all cuts vanish. Similar reasoning shows that the all-plus helicity loop amplitude does not contain multiparticle poles. The only singularities are those where two (color-adjacent) momenta become collinear.

Another simplifying feature of the all-plus amplitude is the equality, up to a sign due to statistics, of the contributions of internal gluons, complex scalars, and Weyl fermions. This is a consequence of the supersymmetry Ward identity [13] $A^{\text{SUSY}}(1^\pm, 2^+, \ldots, n^+) = 0$ for $N = 1$ and $N = 2$ theories. For Weyl fermions and complex scalars transforming under the fundamental rather than the adjoint representation (in a vectorlike theory), the color factor is smaller by a factor of N, and *no* subleading color factors appear.

At one loop the collinear limits of color-ordered one-loop QCD amplitudes are expected to have the form

$$A_{n;1}^{\text{loop}} \xrightarrow{a\|b} \sum_{\lambda=\pm} \{\mathrm{Split}_{-\lambda}^{\text{tree}}(a^{\lambda_a}, b^{\lambda_b}) A_{n-1;1}^{\text{loop}}[\cdots (a+b)^\lambda \cdots] + \mathrm{Split}_{-\lambda}^{\text{loop}}(a^{\lambda_a}, b^{\lambda_b}) A_{n-1}^{\text{tree}}[\cdots (a+b)^\lambda \cdots]\}, \quad (1)$$

in the limit where the momenta $k_a \to zk_P$ and $k_b \to (1-z)k_P$ with $k_P = k_a + k_b$. Here λ is the helicity of the intermediate state with momentum k_P. This is analogous to the form of tree-level collinear limits [1,2,4,14]. The explicit form of the one-loop splitting functions may be extracted from the known four- [15] and five-point [8] gluon amplitudes. All known one-loop amplitudes [8,12] satisfy Eq. (1), though there is as yet no proof of its correctness of larger n. Because of the supersymmetry Ward identity relating the gluon and fermion contribution to the scalar one, it suffices for our present purposes to prove it for the case of scalars in the loop.

The one-loop all-plus helicity amplitudes have a simple collinear structure because the loop splitting function $\mathrm{Split}_{-\lambda}^{\text{loop}}$ does not enter; it multiplies a tree amplitude which vanishes. The tree splitting functions that enter are [1,2,4]

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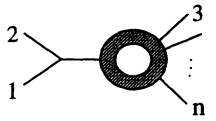

FIG. 1. Diagrams that contribute to the tree splitting functions.

$$\text{Split}_{++}^{\text{tree}}(a^+,b^+)=0\,,$$
$$\text{Split}_{--}^{\text{tree}}(a^+,b^+)=1/[\sqrt{z(1-z)}\langle ab\rangle]\,, \tag{2}$$

where we follow the notation of Ref. [14] for the spinor inner products $\langle ab\rangle$ and $[ab]$ which are equal to $\sqrt{s_{ab}}$ up to a phase. In general, the nonvanishing splitting functions diverge as $1/\sqrt{s_{ab}}$ in the collinear limit $s_{ab}=(k_a+k_b)^2\to 0$.

We now outline a proof of the universality of the scalar-loop contributions to the collinear splitting functions. We divide the diagrams into several sets, depending upon the topology of the two external collinear legs which, without loss of generality, we label 1 and 2. In a color-ordered diagram, only adjacent legs can have collinear singularities. It turns out that $\text{Split}^{\text{tree}}$ arises from the diagrams in Fig. 1, $\text{Split}^{\text{loop}}$ from the diagrams in Fig. 2, and diagrams without explicit poles in s_{12}, such as those in Fig. 3, do not contribute to the splitting functions.

We begin with the diagrams in Fig. 1. The only Feynman diagrams which can contribute to the tree splitting function are those containing explicit poles in s_{12}, as depicted in Fig. 1; trees containing legs 1 and 2 but lacking this explicit pole will not contribute. The analysis is identical to the tree-level analysis and gives a similar result, yielding the first term in Eq. (1) containing the tree splitting function.

The diagrams in Fig. 2 also contain explicit collinear poles and give rise to the $\text{Split}^{\text{loop}}$ function. There are three groups of diagrams in this category depicted in Figs. 2(a)–2(c). Evaluating and summing over the three types of diagrams in the collinear limit yields

$$\frac{1}{16\pi^2}\frac{1}{6}(k_1-k_2)^\mu\eta_{\mu\nu}A_{n-1}^{\text{tree}}(1+2,\ldots,n)^\nu \times\left(\frac{\sqrt{2}}{s_{12}}\right)\left[\varepsilon_1\cdot\varepsilon_2-\frac{\varepsilon_1\cdot k_2\,\varepsilon_2\cdot k_1}{k_1\cdot k_2}\right], \tag{3}$$

where ε_i are gluon polarization vectors. This will give the entire contribution to the loop splitting functions for internal scalars. Converting to a helicity basis [16] in a manner similar to that used at tree level in Ref. [2], one finds

$$\text{Split}_{++}^{\text{loop[0]}}(a^+,b^+)=-\sqrt{z(1-z)}[ab]/(48\pi^2\langle ab\rangle^2)\,,$$
$$\text{Split}_{--}^{\text{loop[0]}}(a^+,b^+)=\sqrt{z(1-z)}/(48\pi^2\langle ab\rangle)\,, \tag{4}$$

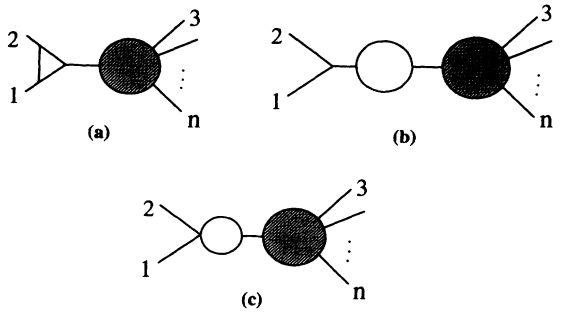

FIG. 2. Diagrams that contribute to the loop splitting functions.

and $\text{Split}_{-\lambda}^{\text{loop[0]}}(a^\pm,b^\mp)$ vanishes.

The remaining diagrams do not have the required collinear pole arising from a tree propagator; it would have to emerge from the loop integral. One possibility is that one collinear leg is directly connected to the loop via a three vertex while the other collinear leg is part of a tree or a four vertex sewn onto the loop. These diagrams cannot have any collinear poles in s_{12} because the loop integral does not contain this kinematic invariant except in a sum with other kinematic invariants.

The next possibility, depicted in Fig. 3(a), is that both legs in the collinear pair are attached to a scalar loop by three-point vertices and are part of a loop with four or more legs. Since the splitting functions diverge as $s_{12}\to 0$, contributions come from regions where the three propagators $1/(l-k_2)^2$, $1/l^2$, and $1/(l+k_1)^2$ depicted in Fig. 3(a) blow up. The leading singularities come from the region $l\approx\alpha k_1+\beta k_2$ where α and β are arbitrary constants. Near the special points $(\alpha,\beta)=(-1,0)$ and $(0,1)$ a fourth propagator blows up requiring a separate analysis, which will lead to the same conclusion as the generic case. In the generic case, in the region $l\approx\alpha k_1+\beta k_2$ the calculation reduces to a triangle integral. An analysis of the integral [17] shows that there are no contributions to the splitting functions from Fig. 3(a) or 3(b). For gluons or fermions circulating in the loop (for a generic helicity amplitude), loop-momentum-independent terms in the vertices of the diagrams in Fig. 3 invalidate the above analysis.

The starting point in constructing our n-point expression is the known five-point one-loop helicity amplitude [8],

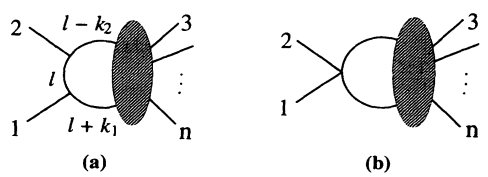

FIG. 3. Two of the remaining diagram types which have no collinear poles for scalars in the loop.

$$A_{5;1}(1^+,2^+,3^+,4^+,5^+)=\frac{iN_p}{192\pi^2}\frac{s_{12}s_{23}+s_{23}s_{34}+s_{34}s_{45}+s_{45}s_{51}+s_{51}s_{12}+\varepsilon(1,2,3,4)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}, \tag{5}$$

where $\varepsilon(i,j,m,n)=4i\varepsilon_{\mu\nu\rho\sigma}k_i^\mu k_j^\nu k_m^\rho k_n^\sigma=[ij]\langle jm\rangle[mn]\langle ni\rangle-\langle ij\rangle[jm]\langle mn\rangle[ni]$, and N_p is the number of color-weighted bosonic states minus fermionic states circulating in the loop; for QCD with n_f quarks, $N_p=2(1-n_f/N)$ with $N=3$.

Using Eqs. (1) and (2) and $A_n^{\rm tree}(1^\pm,2^+,\ldots,n^+)=0$, we can construct higher point amplitudes by writing down general forms with only two particle poles, and requiring that they have the correct collinear limits. Generalizing to all n we have

$$A_{n;1}(1^+,2^+,\ldots,n^+)=\frac{iN_p}{192\pi^2}\frac{E_n+O_n}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}, \tag{6}$$

where

$$O_n=\sum_{1\le i_1<i_2<i_3<i_4\le n-1}\varepsilon(i_1,i_2,i_3,i_4)$$
$$=-\sum_{1\le i_1<i_2<i_3<i_4\le n}\mathrm{tr}(\not k_{i_1}\not k_{i_2}\not k_{i_3}\not k_{i_4}\gamma^5). \tag{7}$$

To describe E_n define $t_i^{[p]}=(k_i+k_{i+1}+\cdots+k_{i+p-1})^2$ (all indices mod n); note that $t_i^{[2]}=s_{i,i+1}$ and $t_i^{[1]}=0$. Then

$$E_n=\sum_{p=2}^{[(n-1)/2]}\sum_{i=1}^{n}(t_i^{[p]}t_{i+1}^{[p]}-t_{i+1}^{[p-1]}t_i^{[p+1]})$$
$$+\left[\frac{1}{2}\sum_{i=1}^{n}(t_i^{[m]}t_{i+1}^{[m]}-t_{i+1}^{[m-1]}t_i^{[m+1]})\right]_{m=n/2,\ n\ \text{even}} \tag{8}$$

or equivalently

$$E_n=-\sum_{1\le i_1<i_2<i_3<i_4\le n}\mathrm{tr}(\not k_{i_1}\not k_{i_2}\not k_{i_3}\not k_{i_4}). \tag{9}$$

The two terms O_n and E_n can be combined into a single trace, a form which agrees with Ref. [6], but for the purposes of discussing symmetry properties, it is more convenient to keep them separate.

The O_n term (7) is not manifestly cyclicly symmetric; however, the difference between O_n and its cyclic permutation vanishes using momentum conservation. To verify that in the limit that two legs become collinear it reduces to the corresponding $(n-1)$-point term O_{n-1}, it suffices to check the limit $1\|2$. Terms of the form $\varepsilon(1,2,j_3,j_4)$ clearly vanish. The remaining terms containing 1 and 2 may be paired as $\varepsilon(1,i_2,i_3,i_4)+\varepsilon(2,i_2,i_3,i_4)=\varepsilon(P,i_2,i_3,i_4)$, where $k_P=k_1+k_2$. Adding these terms to the terms containing neither 1 nor 2, and relabeling $\{P,3,4,\ldots,n\}\to\{1,2,3,\ldots,n-1\}$, we see that $O_n\to O_{n-1}$ in the limit $1\|2$, as required. The cyclic symmetry of the E_n term (8) is manifest. The collinear limit of the equivalent form (9) follows the same argument as for the O_n terms.

Assuming that the denominator of the all-plus amplitude is given by $\langle 12\rangle\cdots\langle n1\rangle$, one can prove that the functions E_n and O_n are uniquely determined by the collinear limits for all $n>5$. [The collinear limit of O_5 is special because $\varepsilon(1,2,3,4)$ vanishes in all collinear limits.] Presumably one should be able to give a proof of the same statement relaxing the denominator assumption.

In massless QED, through use of recursion relations [2,3], Mahlon has demonstrated that the one-loop n-photon helicity amplitudes $A_n(\gamma_1^\pm,\gamma_2^+,\ldots,\gamma_n^+)$ vanish for $n>4$ [18]. One can generalize Mahlon's results in the all-plus case to "mixed" photon-gluon amplitudes using the expression (6) and converting some of the gluons into photons. Amplitudes with r external photons and $(n-r)$ gluons have a color decomposition similar to that of the pure-gluon amplitudes, except that charge matrices are set to unity for the photon legs. The coefficients of these color factors, $A_{n;1}^{r\gamma}$, are given by appropriate cyclic sums over the pure-gluon partial amplitudes, retaining only the contributions from particles in the fundamental representation in the loop; e.g., for a single quark with electric charge Q, replace $N_p\to N_p^{\rm fund}=-2/N$, and the overall coupling factor $g^n\to g^{n-r}(eQ\sqrt{2})^r$. Defining the short-hand $\mathcal{S}_n(i,j)=\langle ij\rangle/(\langle in\rangle\langle nj\rangle)$, performing the cyclic sums, and making repeated use of spinor identities we can write down simple forms for the all-plus partial amplitude with one or two external photons (legs $n\cdots n-r+1$), and any number of gluons,

$$A_{n;1}^{r\gamma}=\frac{iN_p^{\rm fund}}{192\pi^2}\frac{O_n^{r\gamma}+E_n^{r\gamma}}{\langle 12\rangle\langle 23\rangle\cdots\langle n-r-1,n-r\rangle\langle n-r,1\rangle}, \tag{10}$$

with

$$O_n^{1\gamma}=-2\sum_{1\le i_1<i_2<i_3<i_4\le n-1}\varepsilon(i_1,i_2,i_3,i_4)[\mathcal{S}_n(i_1,i_2)+\mathcal{S}_n(i_3,i_4)],$$
$$E_n^{1\gamma}=2\sum_{1\le i_1<i_2<i_3\le n-1}[\mathcal{S}_n(i_1,i_2)s_{i_1i_2}s_{i_3n}+\mathcal{S}_n(i_2,i_3)s_{i_2i_3}s_{i_1n}+\mathcal{S}_n(i_3,i_1)s_{i_3i_1}s_{i_2n}],$$
$$O_n^{2\gamma}=4\sum_{1\le i_1<i_2<i_3\le n-2}\varepsilon(i_1,i_2,i_3,n-1)[\mathcal{S}_n(i_1,i_2)\mathcal{S}_{n-1}(i_2,i_3)-\mathcal{S}_{n-1}(i_1,i_2)\mathcal{S}_n(i_2,i_3)], \tag{11}$$
$$E_n^{2\gamma}=4\sum_{1\le i_1<i_2\le n-2}\left[\mathcal{S}_{n-1}(i_1,i_2)\mathcal{S}_n(i_1,i_2)\mathrm{tr}(\not k_{i_1}\not k_n\not k_{i_2}\not k_{n-1})-s_{i_1i_2}\frac{[n-1n]}{\langle n-1n\rangle}\right].$$

For three external photons, an even more striking result emerges: the amplitude vanishes,

$$A_n^{\text{loop}}(\gamma_1^+,\gamma_2^+,\gamma_3^+,g_4^+,\ldots,g_n^+)=0. \quad (12)$$

Since amplitudes with even more photon legs are obtained by further sums over permutations of legs, the all-plus helicity amplitudes with three or more photon legs vanish (for $n>4$) in agreement with the expectation from the collinear limits.

In order to extend these methods to other helicity amplitudes one would first need a general proof of the collinear limits for particles circulating in the loop other than scalars [17] (which sufficed for the all-plus case because of the supersymmetry identities). The loop splitting functions appearing in Eq. (1) can already be extracted from five-parton amplitudes [8,12]. We expect that collinear limits will be a useful tool in constructing one-loop helicity amplitudes besides those presented here.

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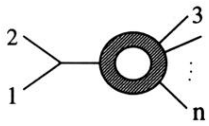


FIG. 1. Diagrams that contribute to the tree splitting functions.

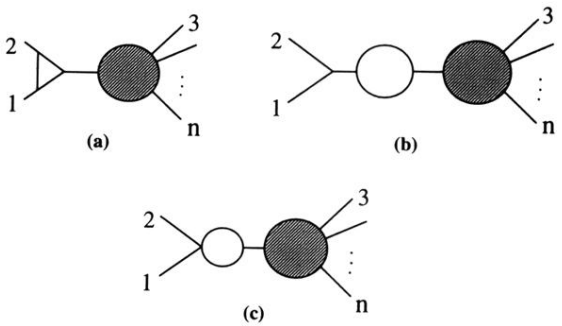
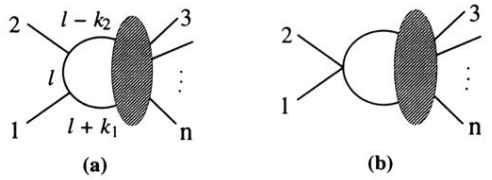


FIG. 2. Diagrams that contribute to the loop splitting functions.



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