Subrecoil Laser Cooling and Lévy Flights

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Anomalous diffusion processes, dominated by rare events, are shown to exist and to play a central role in certain subrecoil laser cooling schemes. We present a new statistical analysis of these processes, in terms of Lévy flights, which provides a precise analytical description of the cooled atoms in the long time limit, where the standard methods of quantum optics are inappropriate. These analytical predictions are quantitatively checked by comparison with the results of quantum Monte Carlo simulations of the cooling process at intermediate times.

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Laser cooling [1] provides interesting examples of random walks. The atomic momentum p changes in a random way, as a result of coherent absorptions and stimulated emissions of laser photons between a ground state manifold (g) and an excited one (e), interrupted at random times by spontaneous emission of fluorescence photons. In usual cooling schemes, the exchanges of momentum between atoms and laser photons give rise to a net drift of the atomic momenta towards $p=0$. Competing with such a friction mechanism, there is also a momentum diffusion due to the randomness introduced by spontaneous emission, leading to a random walk in momentum space, as in Brownian motion. The random recoil $\hbar k$ due to fluorescence photons also gives rise to an important landmark in the temperature scale, the recoil temperature T_R, given by $k_B T_R/2 = E_R = \hbar^2 k^2/2M$.

Up to now, two subrecoil cooling schemes have been demonstrated [2,3]. We will focus here on the first one [2], based on velocity selective coherent population trapping (VSCPT). Some of the ideas developed here could possibly be extended to the other scheme and to other proposals [4]. In VSCPT, there is no friction mechanism, but a combination of two effects. First, there are quantum interference effects which prevent atoms from absorbing light if they are in certain linear superpositions of ground state sublevels (dark states) and if they have a very small momentum $p \simeq 0$. Second, for atoms with $p \neq 0$, which scatter photons in random directions, there is a momentum diffusion in p space which allows some atoms to be transferred from the $p \neq 0$ absorbing states into the $p \simeq 0$ dark states where they remain trapped for a long time, and where they pile up. An important question then arises concerning the ultimate efficiency of such a cooling process. Using simple arguments, we have argued [2] that the width of the momentum distribution has no lower limit and should decrease as $\theta^{-1/2}$, when the laser-atom interaction time θ is increased. However, up to now, no quantitative prediction concerning the proportion of cooled atoms and the exact shape of the momentum distribution was available [5]. This is not surprising in view of the complexity of the full quantum optical Bloch equations (both internal and external degrees of freedom must be treated quantum mechanically at these subrecoil temperatures). Furthermore, in the problem studied here, where no steady state exists, it seems quite difficult to extract the asymptotic behavior from a numerical solution of optical Bloch equations.

In order to improve our understanding of VSCPT, we have performed quantum Monte Carlo simulations [6], using the "delay function" [7]. It clearly appears (see Fig. 1) that the smaller the atomic momentum p, the longer the delay τ_d between two successive spontaneous emissions, which is the principle of VSCPT. There is another striking feature of Fig. 1 which is the starting point of the analysis presented here: The random sequence of time intervals τ_d is clearly dominated by a sin-

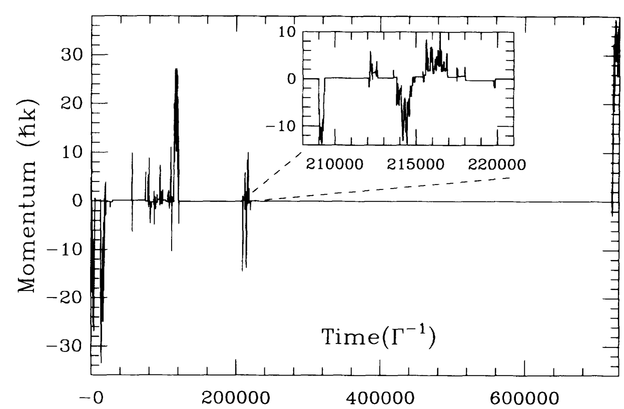

FIG. 1. Monte Carlo simulation of VSCPT. Each vertical discontinuity corresponds to a spontaneous emission jump during which the atomic momentum p changes abruptly. The delay between two successive jumps can become very long if p gets close to zero. The longest time interval (out of 4000) takes about 70% of the total time. The inset shows a zoomed part of the sequence.

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gle term, the longest one, which is on the order of the total observation time. This uncommon domination of a random sequence by rare events is a signature of "Lévy flights" and "broad" distributions [8,9]—in sharp contrast with the usual Brownian motion statistics encountered in other cooling schemes. This feature has led us to develop a completely new quantitative analysis of VSCPT, based purely on statistical arguments. The advantage of such an approach is to provide explicit analytical results for the cooling efficiency at long interaction times.

The basic ingredient of our statistical analysis is the dependence, with the atomic momentum p, of the fluorescence rate $R_F(p)$ from dark states (curve in solid line of Fig. 2). The dip around $p=0$ and the null value of $R_F(p)$ for $p=0$ show that these states become darker and darker when $p \rightarrow 0$ [$R_F(p) \propto p^2$ when $p \rightarrow 0$]. When $|p|$ increases, $R_F(p)$ reaches a value Γ' equal to the absorption rate from g in the absence of quantum interference. Finally, when $|p| > p_e$, the Doppler shift is so large that the lasers get out of resonance with the atomic transition, and $R_F(p)$ decreases. The widths p_g and p_e of the narrow and broad structures of Fig. 2 are such that $kp_g/M \sim \Gamma'$ and $kp_e/M = \Gamma/2$, where Γ is the spontaneous emission rate from e. Although this is not essential, we suppose here that the laser intensities are low enough, so that $\Gamma' \ll \Gamma$, or equivalently $p_g \ll p_e$.

It is now convenient to define in the neighborhood of $p=0$ a narrow trapping zone $|p| \leq p_{\mathrm{trap}}$ (we choose $p_{\mathrm{trap}} < \hbar k, p_g$), where atoms can remain trapped [10] for a very long time $\tau(p)$ on the order of $1/R_F(p) \propto p^{-2}$. The temporal evolution of the atom then appears as a sequence of trapping periods where $|p| \leq p_{\mathrm{trap}}$, with durations $\tau_1, \tau_2, \ldots$, alternating with diffusion periods, with durations $\hat{\tau}_1, \hat{\tau}_2, \ldots$, where the atom, with $|p| > p_{\mathrm{trap}}$, is diffusing out of the trap. The $\hat{\tau}_i$'s are actually "first return times" in the trap. Consider $2N$ successive alternating trapping and diffusion periods, with $N \gg 1$, and let $T(N) = \sum_{i=1}^{N} \tau_i$ be the total trapping time, and $\hat{T}(N) = \sum_{i=1}^{N} \hat{\tau}_i$ the total escape time. We first try to understand how $T(N)$ and $\hat{T}(N)$ grow with N. Since the τ_i's are independent random variables, as well as the $\hat{\tau}_i$'s, we need only to find their probability distributions $P(\tau)$ and $\hat{P}(\hat{\tau})$.

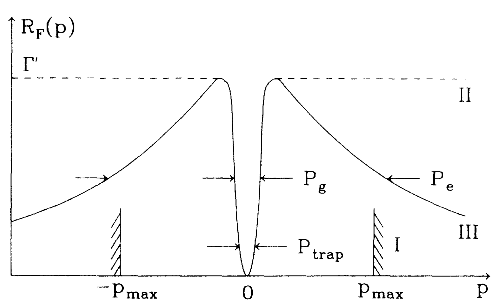

FIG. 2. Variations with p of the fluorescence rate $R_F(p)$ (see text). The narrow dip around $p=0$, with a width p_g, is due to VSCPT. The trapping zone is defined by $|p| < p_{\mathrm{trap}}$. Three different models are taken for the variations of $R_F(p)$ at large p. Model I: walls confining the atomic momentum to $|p| \leq p_{\max}$. Model II: constant fluorescence rate equal to Γ' out of the dip (interrupted line). Model III (corresponding to actual experiments, full line): decrease of the fluorescence rate for $|p| > p_e$, due to a Doppler detuning from the optical resonance.

In order to calculate $P(\tau)$, we first introduce the probability $\Pi(p)$, for an atom falling in the trap, to reach it with a momentum p. Since we have chosen $p_{\mathrm{trap}} < \hbar k$, and since the momentum change after a spontaneous emission process is on the order of $\hbar k$, $\Pi(p)$ is nearly uniform and equal to $\frac{1}{2} p_{\mathrm{trap}}$. It is then easy to calculate the asymptotic form (for large τ) of $P(\tau)$. Using $\tau(p) \propto p^{-2}$ (since $p_{\mathrm{trap}} < p_g$) and $\Pi(p)dp = P(\tau)d\tau$, we find that $P(\tau)$ varies as $\Pi(p)dp/d\tau \propto p^3$, i.e., as $\tau^{-3/2}$. More precise calculations, which will be published elsewhere [11], give $P(\tau) = B\tau^{-3/2}$ with $B = M\sqrt{\pi}/8kp_{\mathrm{trap}}\sqrt{2\tau_0}$ and $\tau_0 = 1/\Gamma'$. Unlike usual Gaussian or even Lorentzian distributions, $P(\tau)$ decreases very slowly at large τ. The distribution of trapping times is thus a broad one, so broad that $\langle \tau \rangle$ is infinite. Consequently, the standard central limit theorem (CLT) does not apply to the sum $T(N)$. It must be replaced by a generalized CLT, established by Lévy and Gnedenko (see, e.g., [9] for a concise account), and dealing with normalized power law distributions $P(\tau)$, behaving at large τ as $B\tau^{-(1+\mu)}$ with $0 < \mu \leq 2$ (here, $\mu = \frac{1}{2}$). One can then show [9] that $T(N)$ is a "Lévy sum" which, for $0 < \mu < 1$, does not grow as N for large N, but rather as $N^{1/\mu}$ (here as N^2), the total sum being dominated by the largest term. More precisely, the probability to find the rescaled variable $x = T(N)/N^{1/\mu}$ ranging between x_1 and x_2 is given by $\int_{x_1}^{x_2} \mathcal{L}_\mu^B(x)dx$, where $\mathcal{L}_\mu^B(x)$ is a "Lévy distribution law" (equal to the inverse Laplace transform of $\exp\{-[\pi B/\mu\Gamma(\mu)\sin(\pi\mu)]s^\mu\}$; see Ref. [9]).

We now switch to $\hat{P}(\hat{\tau})$ and we first consider possible behaviors of $R_F(p)$ at large $|p|$ simpler than the solid line of Fig. 2. If the atomic momentum is confined to $|p| \leq p_{\max}$ (model I), for example, by a friction mechanism operating at large p, there exists a mean value for the first return times $\hat{\tau}_i$, which is on the order of the diffusion time from 0 to $p_{\max}$. Therefore $\hat{T}(N)$ simply varies as N. Another model consists of taking an unconstrained momentum diffusion, with a constant rate Γ', as soon as p is out of the dip of Fig. 2 (model II, interrupted line of Fig. 2). This is the case if, in the real situation, we have very large values for p_e (or Γ), keeping p_g (or Γ') fixed. We also take $\theta \ll \theta_e$ where $\theta_e \sim \tau_0(p_e/\hbar k)^2$, so that the atom has not yet reached the zone $|p| \simeq p_e$ after the interaction time θ. The $\hat{\tau}_i$'s then coincide with the first return times of usual Brownian motion and it is known [9] that $\hat{P}(\hat{\tau})$ also decreases as $\hat{B}_{\mathrm{II}}\hat{\tau}^{-(1+\hat{\mu})}$ with, in one dimension, $\hat{\mu} = \frac{1}{2}$ and $\hat{B}_{\mathrm{II}} = \hbar k/p_{\mathrm{trap}}\sqrt{6\pi\Gamma'}$ [11]. $\hat{T}(N)$ has thus the same N dependence as $T(N)$, i.e., N^2. We finally come back to the more realistic complete p dependence of $R_F(p)$, which is probed by the atom for very long interaction times $\theta > \theta_e$ (model III, full line of Fig. 2). $\hat{P}(\hat{\tau})$ is then even broader than in the previous case, because of the $1/p^2$ Doppler decrease of $R_F(p)$ at large p. In this case, as will be shown in [11], $\hat{P}(\hat{\tau})$ decreases as $\hat{B}_{\rm III}\hat{\tau}^{-(1+\hat{\mu})}$, where $\hat{\mu}=\frac{1}{4}$, $\hat{B}_{\rm III}=\beta\tau_0^{1/2}\theta_e^{-1/4}$, β being a number of order 1. For large N, we thus expect that $\hat{T}(N)$ varies as $N^{1/\hat{\mu}}=N^4$.

From the previous results, one can now simply guess how the cooling efficiency at large θ, or large N [12], is influenced by the boundary conditions. With confining walls (model I), $T(N)\sim N^2$ predominates over $\hat{T}(N)\sim N$, so that atoms will spend most of their time in the trap and will be ultimately all trapped. Without reflecting walls, and for $\theta \ll \theta_e$ (model II), $T(N)$ and $\hat{T}(N)$ both behave as N^2. The proportion of trapped atoms will thus tend to a constant. Finally, for $\theta > \theta_e$ (model III), $\hat{T}(N)\sim N^4$ predominates over $T(N)\sim N^2$, so that atoms will spend most of their time out of the trap.

Up to now, we have considered only 1D problems. Similar arguments can be used at higher dimensions. Assuming that $R_F \propto p^2$ still holds, we have, in d dimensions, $P(\tau)\sim\tau_0^{\mu}\tau^{-(1+\mu)}$ with $\mu=d/2$; $d=2$ hence appears as a marginal case, where $\langle\tau\rangle$ still diverges, although only logarithmically. For $d>2$, $\langle\tau\rangle$ is finite. On the contrary, $\hat{P}(\hat{\tau})$ gets broader. In the absence of Doppler decrease at large $|p|$, one finds that $\hat{P}(\hat{\tau})$ decays only as $(\hat{\tau}\ln^2\hat{\tau})^{-1}$ in $d=2$, and is not well defined when $d>2$ since a finite fraction of the atoms will never come back to their starting point. In this case, both effects act in the same direction to reduce the cooling efficiency as d is increased: $P(\tau)$ narrows, and $\hat{P}(\hat{\tau})$ widens. Of course, as in 1D, confining walls at large p lead to a narrow distribution $\hat{P}(\hat{\tau})$, dramatically improving the cooling efficiency.

The previous analysis shows that the competition between trapping and escape processes is controlled by the exponents μ and $\hat{\mu}$ characterizing $P(\tau)$ and $\hat{P}(\hat{\tau})$. More quantitative results can be derived [11]. We just give here the outline of the calculations. Let f be the probability to find the atom in the trap at the end of the interaction time θ. We have $f=\int_0^\theta \pi(\theta-\tau)\chi(\tau)d\tau$, where $\pi(\theta-\tau)$ is the probability that the last entry of the atom in the trap occurs at time $\theta-\tau$ (after an arbitrary number of trapping and escape periods), and where $\chi(\tau)=\int_\tau^\infty P(\tau')d\tau'$ is the probability that the atom has not left the trap after a time τ. One can show that, for models II and III, $\pi(t)$ can be expressed in terms of the convolution of two Lévy laws with exponents μ and $\hat{\mu}$. If $\mu=\hat{\mu}$, $\pi(t)$ is also given by a Lévy law and behaves at large t as $t^{\mu-1}$. If $\hat{\mu}<\mu$, the broader Lévy law is predominant and $\pi(t)$ behaves asymptotically as $t^{\hat{\mu}-1}+O(t^{(2\hat{\mu}-\mu-1)})$.

The following results can then be derived for f. For model I (confining walls), f tends to 1 at large θ. For model II [no Doppler decrease for $R_F(p)$], f tends asymptotically towards a constant value $f_0=B/(B+\hat{B}_{\rm II})$, or equivalently $f_0=[1+a(E_R/\hbar\Gamma')]^{-1}$ where $a=16/\pi\sqrt{3}$ [5]. Finally, for model III [Doppler decrease for $R_F(p)$], and for $\theta>\theta_e$, we have $f^{-1}=A(\theta/\theta_e)^{1/4}+K(\theta)$, where $K(\theta)$ is a slowly (logarithmically) varying function of θ. Exact expressions for the prefactor A and for $K(\theta)$ will be given in [11].

We have compared our predictions at 1D for the proportion f of cooled atoms with the results of Monte Carlo simulations using the delay function (Fig. 3). For large θ, it is more efficient numerically to use such an approach than to integrate optical Bloch equations, as in Ref. [2]. We have checked that, for model I, f tends to 1. For model II, the numerical experiment (squares) confirms that f tends to a constant which agrees quite well with the predicted value (interrupted line). As for model III, we check that the decrease of f that we predict here for $\theta>\theta_e$ actually occurs. For numerical reasons, the time range investigated cannot be extremely large compared to θ_e, and the slowly varying function $K(\theta)$ cannot be tested. So, we have fitted our numerical data (circles) with the two parameter formula $f^{-1}=A_{\rm num}(\theta/\theta_e)^{1/4}+\bar{K}$, where the constant $\bar{K}$ mimics all the subleading terms. As can be seen in Fig. 3, this form (full line) fits our data extremely well. The value found for $A_{\rm num}$ also has the right order of magnitude.

The Lévy flights analysis can also yield the momentum distribution $\rho(p)$ of trapped atoms [11]. One finds that, in agreement with the simple predictions of [2], the width of $\rho(p)$ decreases in 1D as $\theta^{-1/2}$, with no lower limit, when θ is increased. Such a result is extended to 2D and 3D. The wings of $\rho(p)$ are also found to decrease in the

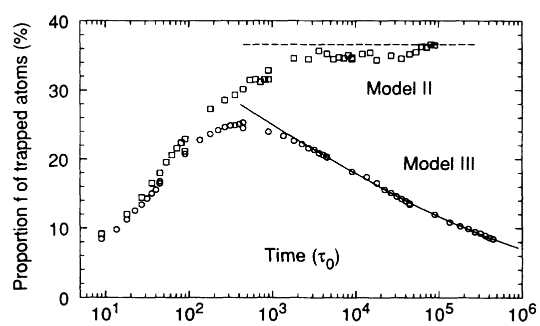

FIG. 3. Variations with the interaction time θ of the proportion f of trapped atoms ($|p|<p_{\rm trap}$), calculated from N Monte Carlo runs for the models II and III of Fig. 2. Model II (squares): $N=4000$, $p_g=0.5\hbar k$, $p_{\rm trap}=0.08\hbar k$; the interrupted line represents the asymptotic theoretical prediction $f=0.365$ corresponding to $E_R/\hbar\Gamma'=0.59$. Model III (circles): $N=16000$, $p_g=0.5\hbar k$, $p_{\rm trap}=0.08\hbar k$, $p_e=9.4\hbar k$; the full line represents the best fit for the asymptotic theoretical prediction (see text). Model II requires more computer time than model III. This is why N is smaller and the statistical uncertainty larger.

trapping zone as $1/p^2$, i.e., more slowly than for a Maxwell distribution. Note that the momentum distribution explicitly depends on the interaction time θ. Such a nonstationary behavior, which is typical of Lévy processes, is a signature of "weak ergodicity breaking" [13]. Ergodicity requires the possibility to average the evolution of a single atom over an interaction time θ long compared to any characteristic time. But here, as long as θ can be, there are always characteristic times (trapping times) which can be longer than θ, provided that p is small enough.

In conclusion, we have established fruitful connections between two different fields: laser cooling of atoms to ultralow temperatures, on the one hand and statistical description of anomalous diffusion processes, on the other hand. Laser cooling can provide simple and realistic models of random processes with unusual features. Reciprocally, in the laser cooling field, where one tries to push the limits farther and farther, and where, correspondingly, detailed microscopic calculations become more and more intricate, global approaches, such as purely statistical ones, seem to us very useful. For example, we have established here for the first time that the proportion f of cooled atoms should, at 1D, decrease as $\theta^{-1/4}$ at long interaction times θ. The shape of the full momentum distribution $\rho(p)$ can also be investigated. In our new approach, the emphasis is shifted from optical Bloch equations to the much simpler determination of the asymptotic behavior of the distributions of trapping times and first return times, which contain the relevant parameters controlling the cooling efficiency [14].

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[1] *Laser Cooling and Trapping of Atoms*, Special Issue edited by S. Chu and C. Wieman [J. Opt. Soc. Am. B **6**, 2019 (1989)]; *Laser Manipulation of Atoms and Ions*, Proceedings of the 1991 Varenna Summer School, edited by E. Arimondo, W. D. Phillips, and F. Strumia (North-Holland, Amsterdam, 1992).

[2] A. Aspect, E. Arimondo, R. Kaiser, N. Vansteenkiste, and C. Cohen-Tannoudji, Phys. Rev. Lett. **61**, 826 (1988); J. Opt. Soc. Am. B **6**, 2112 (1989). For an extension of VSCPT to higher dimensions, see M. A. Ol'shanii and V. G. Minogin, in *Proceedings of Light Induced Kinetic Effects*, edited by L. Moi *et al.* (ETS Editrice, Pisa, 1991); F. Mauri and E. Arimondo, Europhys. Lett. **16**, 717 (1991).

[3] M. Kasevich and S. Chu, Phys. Rev. Lett. **69**, 1741 (1992).

[4] D. E. Pritchard, K. Helmerson, V. S. Bagnato, G. P. Lafyatis, and A. G. Martin, in *Laser Spectroscopy VIII*, edited by W. Persson and S. Svanberg (Springer-Verlag, Berlin, 1987). See also H. Wallis and W. Ertmer, J. Opt. Soc. Am. B **6**, 2211 (1989); K. Molmer, Phys. Rev. Lett. **66**, 2301 (1991).

[5] See, however, V.A. Alekseev and D. D. Krylova, Pis'ma Zh. Eksp. Teor. Fiz. **55**, 321 (1992) [JETP Lett. **55**, 321 (1992)]; Laser Phys. **2**, 781 (1992). These authors, using optical Bloch equations, predict that the proportion of cooled atoms should reach an asymptotic value when $\theta \to \infty$, coinciding with the result of our model II defined below. However, this result turns out to be incorrect when the Doppler effect is correctly taken into account (as we do in the model III defined below).

[6] C. Cohen-Tannoudji, F. Bardou, and A. Aspect, in *Laser Spectroscopy X*, edited by M. Ducloy, E. Giacobino, and G. Camy (World Scientific, Singapore, 1992).

[7] C. Cohen-Tannoudji and J. Dalibard, Europhys. Lett. **1**, 441 (1986); P. Zoller, M. Marte, and D. F. Walls, Phys. Rev. A **35**, 198 (1987).

[8] E. W. Montroll and M. F. Schlesinger, in *Studies in Statistical Mechanics*, edited by J. Lebowitz and E. W. Montroll (North-Holland, Amsterdam, 1984); M. F. Schlesinger, G. M. Zaslavsky, and J. Klafter, Nature (London) **363**, 31 (1993).

[9] J. P. Bouchaud and A. Georges, Phys. Rep. **195**, 125 (1990). See also the experiment described in A. Ott, J. P. Bouchaud, D. Langevin, and W. Urbach, Phys. Rev. Lett. **65**, 2201 (1990). The denomination Lévy flight usually refers to processes which are spatially superdiffusive. In this paper, we use it for flights which take place in the time domain.

[10] For long enough interaction times θ, the width of the momentum distribution of trapped atoms will always be smaller than p_{trap}, and the theoretical predictions for the proportion of trapped atoms will no longer depend on the precise value of p_{trap}, which can thus be somewhat arbitrarily chosen.

[11] F. Bardou, J. P. Bouchaud, A. Aspect, and C. Cohen-Tannoudji (to be published).

[12] Throughout this paper, we consider values of θ large enough so that the width of the initial momentum distribution is irrelevant. More precisely, we suppose that most atoms return several times in the trap.

[13] J. P. Bouchaud, J. Phys. I (France) **2**, 1705 (1992).

[14] Suppose, for example, that the cooling scheme is modified in such a way that $R_F(p)$ behaves as p^4 (instead of p^2) near $p=0$. It is then straightforward to show that $\mu = \frac{1}{4} = \hat{\mu}$, so that the cooling efficiency f tends to a constant when $\theta \to \infty$. Correspondingly, the width of $\rho(p)$ decreases as $\theta^{-1/4}$ (rather than as $\theta^{-1/2}$).