# Order $m\alpha^8$ Contributions to the Decay Rate of Orthopositronium

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(Received 28 September 1993)

We discuss how contributions to the order $\mathcal{O}(m\alpha^8)$ orthopositronium decay rate can be separated into two categories, one due to relativistic momenta and calculable in terms of quantum electrodynamics scattering amplitudes, the other due to low momenta and calculable in the simpler framework of a low-energy effective theory. We report new results for all low-momentum contributions, and give a formula relating the remaining contributions to conventional (on-shell) quantum electrodynamics scattering amplitudes.

PACS numbers: 36.10.Dr, 11.10.St, 12.20.Ds

Despite its many successes, quantum electrodynamics (QED) has yet to account fully for the decay rate of orthopositronium. The theoretical expression for this rate is [1,2]

$$\lambda_{\rm th} = \frac{\alpha^6 mc^2}{\hbar}\frac{(2\pi^2-18)}{9\pi}\left[1 - 10.282(3)\left(\frac{\alpha}{\pi}\right) + \frac{1}{3}\alpha^2\,\ln\alpha + B\left(\frac{\alpha}{\pi}\right)^2 - \frac{3}{2\pi}\alpha^3(\ln\alpha)^2\cdots\right], \qquad (1)$$

where the coefficient $B$ has not yet been computed. The measured rate is $\lambda_{\rm exp} = 7.0482(16)\ \mu{\rm s}^{-1}$ [3]. The difference between this value and the known part of $\lambda_{\rm th}$,

$$\lambda_{\rm exp} - \lambda_{\rm th}(B=0) \;=\; 99(16)\times 10^{-4}\ \mu{\rm s}^{-1}, \qquad (2)$$

is surprisingly large: The coefficient $B$ in $\lambda_{\rm th}$ would have to be about 250 to bring theory and experiment into agreement. Such a large coefficient would be unusual but is by no means impossible, particularly given the big $\mathcal{O}(\alpha/\pi)$ correction. A complete calculation of $B$ is essential before any realistic assessment of the situation is possible.

A calculation of these coefficients using traditional bound-state methods is very complicated. This is because each term in a traditional expansion has contributions from both nonrelativistic and relativistic momenta. This means that approximations only valid for small $p$ or only for large $p$ cannot be readily employed to simplify the analysis. In this paper, we outline a new and simpler procedure for computing $B$. Our analysis is based upon a rigorous nonrelativistic reformulation of QED called nonrelativistic quantum electrodynamics (NRQED) [4]. Using this effective field theory, we are able to separate the $\mathcal{O}(\alpha^2)$ corrections into three parts. Two of these involve soft, nonrelativistic momenta and probe the bound-state nature of the system. The other part involves hard, relativistic momenta and is therefore largely insensitive to the details of binding. We have calculated the nonrelativistic contribution and present the results here. We also show how to extract the relativistic contribution from a calculation of ordinary scattering amplitudes; no bound-state physics is required in this part of the calculation.

Our positronium results also have implications for quarkonium decays. These will be discussed in another paper [5].

The lowest-order decay rate of orthopositronium ($o$-Ps, $n=J=S=1$) is given by (we now use natural units, with $c=\hbar=1$)

$$\Gamma_0(o\text{-Ps}\to 3\gamma) = |\Psi(0)|^2\,\hat\sigma_0(0) = \frac{2\pi^2-18}{9\pi}\,m_e\alpha^6 = 7.2112\,\mu{\rm s}^{-1}. \qquad (3)$$

Here $\Psi(0)$ is the ground state Schrödinger-Coulomb wave function evaluated at $\mathbf{r}=0$, and $\hat\sigma_0(p)$ is proportional to the lowest-order decay rate of a *free* electron and positron in an $S$ state:

$$\hat\sigma_0(p) \equiv \frac{1}{4m_e^2}\,{\rm Im}\,\mathcal{M}_0^{(l=0)}\left(e\bar e\to 3\gamma\to e\bar e\right), \qquad (4)$$

where $p$ is the magnitude of the electron center-of-mass momentum.

To order $\mathcal{O}(\alpha^2)$, there are three sources of corrections.

*(1) Radiative corrections to $\hat\sigma_0(0)$.*—It is straightforward to compute the radiative corrections to the on-shell amplitude $\hat\sigma_0(0)$ in both QED and NRQED. As expected, the two theories give different answers for loop contributions from large momenta. Thus the leading NRQED contribution must be renormalized so that the total (tree plus loops) NRQED cross section is the same as in QED. This implies a renormalization of the lowest order decay rate of the form

$$\delta\Gamma_1 \;=\; \delta Z\;\Gamma_0(o\text{-Ps}\to 3\gamma), \qquad (5)$$

where

$$\hat\sigma^l(0) - \hat\sigma^l_{\rm NRQED}(0) \;=\; \delta Z\,\hat\sigma_0(0). \qquad (6)$$



0031-9007/94/72(13)/2006(3)$06.00

Here $\hat{\sigma}^l(0)$ and $\hat{\sigma}^l_{\rm NRQED}(0)$ are the loop contributions to the on-shell amplitudes evaluated at threshold in QED and NRQED, respectively.

We use a photon mass to regulate the infrared (threshold) singularities and choose the UV regulator in NRQED such that

$$\frac{\hat{\sigma}^l_{\rm NRQED}(0)}{\hat{\sigma}_0(0)} = \frac{m_e\alpha}{\lambda} + (2\ln 2+1)\alpha^2\frac{m_e^2}{\lambda^2} -2\times 10.282(3)\frac{\alpha^2}{\pi}\frac{m_e}{\lambda} +\frac{1}{3}\alpha^2\ln\frac{\lambda}{m_e} + \mathcal{O}(\alpha^3). \qquad (7)$$

Since NRQED is defined so that it reproduces QED at low momentum, all infrared divergences cancel in Eq. (6), and $\delta Z$ has a $\lambda$-independent expansion in powers of $\alpha/\pi$:

$$\delta Z = c_1\frac{\alpha}{\pi} + c_2\left(\frac{\alpha}{\pi}\right)^2 + \cdots. \qquad (8)$$

The only $\mathcal{O}(\alpha)$ correction to the decay rate comes from $\delta Z$, and, thus, from Eq. (1) $c_1 = -10.282(3)$. The second-order coefficient has not yet been calculated. The large size of $c_1$ suggests that $c_2$ might also be large. Indeed, a small (gauge invariant) subset of the diagrams contributing to $c_2$ has recently been evaluated and found to contribute 28.8(2) to $c_2$ [6].

*(2) Momentum dependence of $\hat{\sigma}_0(p)$.*—Near threshold, the decay rate for a free electron and positron in an $S$ wave has the form

$$\hat{\sigma}_0(p) = \hat{\sigma}_0(0) + \delta\hat{\sigma}_0(p) + \mathcal{O}\left(\frac{p^4}{m_e^4}\hat{\sigma}_0\right), \qquad (9)$$

where $\delta\hat{\sigma}_0(p)$ can be computed to be

$$\delta\hat{\sigma}_0(p) = -\frac{p^2}{m_e^2}\,\frac{19\pi^2-132}{12(\pi^2-9)}\,\hat{\sigma}_0(0). \qquad (10)$$

This correction term shifts the decay rate by

$$2\Psi(0)\int\frac{d^3p}{(2\pi)^3}\,\delta\hat{\sigma}_0(p)\,\Psi(p). \qquad (11)$$

However, this integral is linearly divergent and must be regulated. We may choose the regulator so that

$$\delta\Gamma_2 = -\frac{E_{o\text{-Ps}}}{m_e}\,\frac{19\pi^2-132}{12(\pi^2-9)}\,\Gamma_0(o\text{-Ps}) \approx 5\times 10^{-4}\ \mu{\rm s}^{-1}, \qquad (12)$$

where $E_{o\text{-Ps}} = -\alpha^2 m_e/4$. (This result applies to any $S$ state provided $E_{o\text{-Ps}}$ is replaced by the energy of the state.) This choice of regulator is such that the Coulombic corrections to $\delta\sigma_0(p)$ do not contribute to $\hat{\sigma}^l_{\rm NRQED}(0)/\hat{\sigma}_0(0)$, Eq. (7).

In NRQED, the decay rate to this order arises from two separate NRQED interactions: $\psi^\dagger\boldsymbol{\sigma}\chi\cdot\chi^\dagger\boldsymbol{\sigma}\psi$ and $\psi^\dagger\boldsymbol{D}^2\boldsymbol{\sigma}\chi\cdot\chi^\dagger\boldsymbol{\sigma}\psi$+H.c., where $\psi$ and $\chi$ are two-component nonrelativistic spinors for the electron and positron, and $\boldsymbol{D}$ is the gauge covariant derivative. We need two renormalization conditions to define these operators. In our calculation, these conditions are given by the definitions Eqs. (7) and (12). These two choices fix uniquely the rest of the computation. (An explicit derivation of the NRQED expansion to second order in $p/m$ and $\alpha$, including all counterterms, will be presented in another paper [7].)

*(3) Corrections to $\Psi(0)$.*—The wave function at the origin in the lowest-order rate [Eq. (3)] is modified by the various relativistic corrections that must be added to the Coulomb-Schrödinger theory. Using first-order perturbation theory, the correction is

$$\delta\Psi(0) = \sum_{n\neq o\text{-Ps}}\Psi_n(0)\frac{\langle n|\,\delta V\,|o\text{-Ps}\rangle}{E_{o\text{-Ps}}-E_n}, \qquad (13)$$

where the sum is over all eigenstates of the unperturbed theory, and

$$\delta V(\mathbf{p},\mathbf{q}) = -(2\pi)^3\delta(\mathbf{p}-\mathbf{q})\frac{p^4}{4m_e^3} - \frac{e^2}{m_e^2q^2}\left(p^2-\frac{(\mathbf{p}\cdot\mathbf{q})^2}{q^2}\right) + \frac{3e^2}{4m_e^2} - \frac{e^2}{4m_e^2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + \frac{e^2}{4m_e^2}\frac{(\mathbf{q}\cdot\boldsymbol{\sigma}_1)(\mathbf{q}\cdot\boldsymbol{\sigma}_2)}{q^2} -3\frac{\alpha^2}{m_e^2}\ln(\mathbf{q}^2/m_e^2). \qquad (14)$$

The potential $\delta V$ contains all $\mathcal{O}(v^2/c^2)$ corrections to the Schrödinger theory as well as the leading $\mathcal{O}(\alpha v^2/c^2)$ correction, given by the $\ln(\mathbf{q}^2/m_e^2)$ term. This last term gives the leading contribution to the Lamb shift in positronium.

The corrections to the wave function will shift the decay rate by

$$\delta\Gamma_3(o\text{-Ps}\to 3\gamma) = 2\,|\delta\Psi(0)\,\Psi(0)|\,\hat{\sigma}_0(0). \qquad (15)$$

As in the case of $\delta\hat{\sigma}(p)$, the ultraviolet divergences encountered in evaluating $\delta\Psi(0)$ are systematically removed by using NRQED. The remaining contributions all come from nonrelativistic loop momenta. As mentioned above, the UV regulators have already been fixed during the computation of $\delta\Gamma_1$ and $\delta\Gamma_2$ so that there is no freedom in the evaluation of these contributions. We find

$$\delta\Psi(0) = \left[\frac{1}{6}\alpha^2 \ln\alpha + 0.58\alpha^2 - \frac{3}{4\pi}\alpha^3(\ln\alpha)^2\right] \Psi(0), \tag{16}$$

which implies

$$\delta\Gamma_3 = \left[\frac{1}{3}\alpha^2 \ln\alpha + 1.16\,\alpha^2 - \frac{3}{2\pi}\alpha^3(\ln\alpha)^2\right] \Gamma_0(o\text{-Ps} \to 3\gamma). \tag{17}$$

The logarithmic terms agree with the literature [1,2]; the other correction is new.

By combining the $\mathcal{O}(\alpha^2\Gamma_0)$ contributions from all three of our corrections, we obtain the final result

$$\begin{aligned}\delta\Gamma_1 + \delta\Gamma_2 + \delta\Gamma_3|_{\alpha^2} &= \left\{\frac{19\pi^2 - 132}{12(\pi^2-9)}\frac{\alpha^2}{4} + 1.16\,\alpha^2 + c_2\left(\frac{\alpha}{\pi}\right)^2\right\}\Gamma_0 \\ &= 9.56\times10^{-4}\,\mu\text{s}^{-1} + c_2(0.39\times10^{-4}\,\mu\text{s}^{-1}).\end{aligned} \tag{18}$$

The coefficient $c_2$ is specified by the threshold rate for $e\bar{e} \to 3\gamma$ [Eqs. (6)–(8)]. The known part of $c_2$ [6] contributes $11\times10^{-4}\ \mu\text{s}^{-1}$ to the rate.

In this paper, we have outlined a new procedure for analyzing the $o$-Ps decay rate at $\mathcal{O}(\alpha^2\Gamma_0)$. We have computed all corrections in this order that depend in detail on bound-state physics. The only remaining contribution [which comes from the coefficient $c_2$ defined in Eqs. (6)–(8)] can be extracted from a calculation of the annihilation rate for a free electron and positron—no bound-state physics is required. Our corrections combined with the known part of the $c_2$ correction account for 20% of the difference between theory and experiment. The unexpectedly large size of these corrections makes it plausible that the full calculation of $c_2$ will bring theory into agreement with experiment.

U.M. thanks the Academy of Finland for support and Cornell for hospitality during the completion of this work. P.L. has been supported by a NSERC fellowship (Canada) and by the Fonds Canadiens à la recherche (Québec). This work was also supported by a grant from the National Science Foundation. We thank Todd Ward for helpful comments concerning the analytical calculation of the $p^2/m_e^2$ correction.

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