Comment on "Spanning Probability in 2D Percolation"

Recently, Ziff [1] calculated the probability $R_L(p)$ for a nearest neighbor (nn) site percolation cluster to span a square lattice of size L at occupancy p, using rule $\mathcal{R}_1$ of Ref. [2] (spanning in one given direction, free boundaries in the other direction). For p near the threshold p_c, and for $L \to \infty$, he fitted the results to the form $R_L(p) = f_1(x) + L^{-\vartheta} f_2(x) + \cdots$, where $x = (p-p_c)L^{1/\nu}$ and $\vartheta = 1$. He found that $f_1(0) = 0.5$, and stated that (1) the latter result contradicts the renormalization group (RG), which would yield $R_L(p_c) \to p_c$ as $L \to \infty$, (2) the function $f_1(x)$ is universal for all 2D systems satisfying $\mathcal{R}_1$, (3) $\vartheta = 1$, and (4) $f_1'(x)$ is even, but $f_2(x)$ has both even and odd terms. The present Comment addresses these statements.

The real space RG of Ref. [2], based on the recursion relation $p' = R_b(p)$, with the length rescale factor b, is only *approximate*. In fact, iterations generate new variables (e.g., bond occupation [3]), and yield flow in a large parameter space. p_c corresponds to the value of p which sits on the critical surface within this space, and flows to the true fixed point. Near the fixed point, R_L depends on the scaling variables, including $t = p - p_c$ and the irrelevant variables ω_i, with recursion relations $t' = b^{1/\nu}t$, $\omega_i' = b^{-\vartheta_i}\omega_i$. Ignoring transient steps, very close to p_c we may iterate until $L = b^l$ [3], $R_L(p) = F(At, B_i\omega_i, L) = F(AtL^{1/\nu}, B_i\omega_i L^{-\vartheta_i}, 1) \equiv \hat{F}(\hat{x}, \hat{y}_i)$. Except for the nonuniversal scale factors A and B_i [4], $\hat{F}(\hat{x}, \hat{y}_i)$ is determined by the fixed point, and is universal. Thus, $R_\infty(p_c) = \hat{F}(0,0)$ depends only on the spanning rule, and is universal. However, $R_\infty(p_c)$ need not be equal to p_c. To set A and B_i one may normalize $\hat{F}$ with some convention, e.g., $\frac{\partial \hat{F}}{\partial \hat{x}} = \frac{\partial \hat{F}}{\partial \hat{y}_i} = 1$ at $(0,0)$.

For two complementary lattices, e.g., the square lattices with nn and nn+nnn (next nn) connectivity, one has [2] $R_L(p) + R_L'(1-p) = 1$. Since $p_c' = 1 - p_c$, this yields $R_\infty(p_c) = \hat{F}(0,0) = 1/2$ [5]. This also implies that $A = A'$, $B_i = -B_i'$, and that $\hat{F}(\hat{x}, \hat{y}_i) - 1/2$ is odd in both $\hat{x}$ and $\hat{y}_i$. Expanding near $(0,0)$,

$$R_L(p) \approx \frac{1}{2} + \hat{f}_1(\hat{x}) + \sum_i \hat{y}_i \hat{f}_{2i}(\hat{x}) + \sum_{i,j} \hat{y}_i \hat{y}_j \hat{f}_{3ij}(\hat{x}), \quad (1)$$

where the universal functions $\hat{f}_1$ and $\hat{f}_{3ij}$ are odd while $\hat{f}_{2i}$ is even in $\hat{x}$.

Concerning Ziff's points, we conclude that (1) although the result $R_\infty(p_c) = 0.5$ contradicts the *approximate* RG of Ref. [2], it is consistent with the *true* RG. (2) Basically, Ziff's novel discussion of universality is correct. However, his function $f_1(x)$ is not universal unless x is replaced by $\hat{x} = Ax$. (3) In addition to $\vartheta_2 = 1$ one should also expect nonanalytic corrections. For 2D percolation, the leading correction has $\vartheta_1 \approx 0.85$ [6]. (4) In contrast to Ziff's Eq. (7), $f_2(x)$ contains only even powers of x.

Equation (1) yields several new predictions. First, as $L \to \infty$ we can ignore the irrelevant variables and expand

$$R_\infty(p) \approx 1/2 + a_1 x + a_3 x^3 + \cdots. \quad (2)$$

However, $a_1 = A\frac{\partial \hat{F}}{\partial \hat{x}}(0,0)$ and $a_3 = A^3 \frac{\partial^3 \hat{F}}{\partial \hat{x}^3}(0,0)/6$ are not universal, whereas a_3/a_1^3 is. Second, using only ϑ_1 and ϑ_2 the data at p_c should scale as

$$R_L(p_c) = 1/2 + \sum_{k,l=1}^{\infty} b_{kl} L^{-k\vartheta_1 - l\vartheta_2}, \quad (3)$$

where $k + l$ is odd. Although b_{kl} are nonuniversal, the combinations $b_{kl}/b_{10}^k b_{01}^l$ are universal. Third, the value $p_{0.5}$, which solves $R_L(p_{0.5}) = 0.5$, scales as $p_{0.5}(L) - p_c \sim (b_{10}L^{-\vartheta_1} + b_{01}L^{-\vartheta_2})L^{-1/\nu}$. Since $\vartheta_1 < 1$, for large L this is dominated by the first term, and not by Ziff's Eq. (1).

To check our predictions, we measured $R_L(p)$ for square lattices with (a) nn, (b) nn+nnn, and (c) nn+nnn+nnnn (3rd nn). Fits by Eq. (2) yielded (a) $a_{1,3} = 0.760 \pm 0.005, -0.455 \pm 0.02$, (b) $a_{1,3} = 0.760 \pm 0.005, -0.45 \pm 0.02$, and (c) $a_{1,3} = 0.845 \pm 0.010, -0.615 \pm 0.03$ (we find $p_c = 0.2891 \pm 0.0002$ [7]). Indeed, $a_3/a_1^3 \approx -1.02 \pm 0.02$ for all cases. (a) and (b) confirm $A = A'$. The analysis of Eq. (3) is more difficult, apparently due to competing signs of the 4 terms with $k + l = 3$. Preliminary fits to our sq(nn) data yielded $b_{10} < b_{01}$, possibly explaining the dominance of $\vartheta = 1$ over Ziff's finite L range. Larger L's are needed to settle this issue.

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Amnon Aharony[1,2] and J-P. Hovi[3]
[1]Department of Physics
University of Oslo, Oslo, Norway
[2]School of Physics and Astronomy, Tel Aviv University
Tel Aviv 69978, Israel
[3]Laboratory of Physics, Helsinki University of Technology, 02150 Espoo, Finland

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