Classification of Quantum Hall Universality Classes by $W_{1+\infty}$ Symmetry

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We show how two-dimensional incompressible quantum fluids and their excitations can be viewed as $W_{1+\infty}$ edge conformal field theories, thereby providing an algebraic characterization of incompressibility. The Kac-Radul representation theory of the $W_{1+\infty}$ algebra leads then to a purely algebraic complete classification of hierarchical quantum Hall states, which encompasses all measured fractions. Spin-polarized electrons in single-layer devices can only have Abelian anyon excitations.

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The central idea in Laughlin's theory [1] of the quantum Hall effect [2] is the existence of *two-dimensional incompressible quantum fluids* at specific rational values $\rho = \nu eB/hc$ of the electron density (B being the external magnetic field). These are macroscopical quantum states with uniform density which are rendered particularly stable by an energy gap. We can think of them semiclassically as *droplets of liquid* without density waves. The absence of density waves causes the longitudinal conductivity to vanish while the Hall conduction is realized as an overall rigid translation of the droplet, giving the Hall conductivity $\sigma_{xy} = \nu e^2/h$.

Our program has been to characterize this picture by a symmetry principle and derive accordingly the *effective field theory* for low energy excitations. Indeed, the *universality* observed in experiments calls for an effective theory approach at long distances, while the extreme precision of the rational values of ν suggests that dynamics is constrained by *symmetry*. Both facts suggest an analogy with two-dimensional critical phenomena, which are classified by conformal field theories [3].

In earlier works [4], we already identified the natural symmetry of *classical* incompressible fluids under *area-preserving diffeomorphisms*, which obey the w_∞ algebra [5]. Actually, these transformations relate all possible configurations of droplets of incompressible fluid by shape changes at constant density. Infinitesimal changes yield *edge excitations* of the droplet [6], whose quantization leads to a (1+1)-dimensional effective conformal field theory defined on the edge [7]. Moreover, the Laughlin quasiparticle excitations with *fractional spin and statistics (anyons)* correspond to fractionally charged edge excitations. These are the relevant low-energy degrees of freedom. In a number of physically relevant examples [4,7], we showed that they span highest-weight representations of the algebra $W_{1+\infty}$ of *quantum area-preserving diffeomorphisms* in the conformal theory. $W_{1+\infty}$ is therefore the symmetry algebra of the low-energy effective theory. Here we use the recently developed [8] representation theory of $W_{1+\infty}$ to classify all possible $W_{1+\infty}$ conformal field theories on the edge of the droplet. This construction provides a complete, purely algebraic classification of all universality classes of incompressible quantum fluids and leads to the identification of the quantum numbers of the excitations.

We start by recalling the *area-preserving diffeomorphisms* of the two-dimensional plane $(z = x + iy,\ \bar{z} = x - iy)$ [4]. Infinitesimal transformations of the distribution function $\rho(z, \bar{z})$ of the fluid are generated by the complete basis $\mathcal{L}^{(cl)}_{n,m} = \rho_0^{(n+m)/2} z^n \bar{z}^m$ via the Poisson brackets $\{f, g\}_{\rm PB} \equiv (i/\rho_0)\left(\partial f \bar\partial g - \bar\partial f \partial g\right)$, with $\partial \equiv \partial/\partial z$, $\bar\partial \equiv \partial/\partial \bar z$, and ρ_0 the average density. The generators $\mathcal{L}^{(cl)}_{n,m}$ satisfy the classical algebra

$$\{\mathcal{L}^{(cl)}_{n,m}, \mathcal{L}^{(cl)}_{k,l}\}_{\rm PB} = -i\,(mk - nl)\,\mathcal{L}^{(cl)}_{n+k-1,m+l-1}\ , \qquad (1)$$

which is usually denoted [5] by w_∞. All classical "small excitations" around a fluid ground state describing a droplet with uniform density ρ_0 are given by the basis $\delta\rho = c_{n,m}\{\mathcal{L}^{(cl)}_{n,m}, \rho\}_{\rm PB}$. As can be checked explicitly [4], they correspond to density waves localized on the sharp boundary of the classical droplet.

Given this fundamental role of w_∞ in the description of classical incompressible fluids, it is to be expected that its *quantum extension* $W_{1+\infty}$, obtained by the substitution $i\hbar\{\ ,\ \}_{\rm PB} \to [\ ,\ ]$ $(z \to z, \bar z \to \partial)$, plays a corresponding role in the physics of *quantum incompressible fluids*.

That this is indeed so can be checked explicitly for the simplest example of a quantum incompressible fluid, the fully filled Landau level [4,9]. As was pointed out by Stone [10], the incompressibility of this state follows solely from the fact that it behaves as a filled Fermi sea in the coordinate plane of the electrons (as opposed to momentum space).

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The field operator for spin-polarized nonrelativistic planar electrons subject to the uniform magnetic field and confined to the first Landau level is given in the symmetric gauge $A_i = B/2\epsilon^{ij}x^j$ by $\Psi(\mathbf{x}) = \sum_{k=0}^{\infty} a_k\, \psi_k(\mathbf{x})$, where

$$\psi_k(z,\bar{z}) = \frac{1}{\ell\sqrt{\pi}}\frac{1}{\sqrt{k!}}\left(\frac{z}{\ell}\right)^k \exp\left(-\frac{|z|^2}{2\ell^2}\right) , \qquad (2)$$

and we denote by $\ell = \sqrt{2/eB}$ the magnetic length (we use units $\hbar = c = 1$). The coefficients in the above expansion are fermionic Fock annihilators satisfying $\left\{a_k, a_l^\dagger\right\} = \delta_{kl}$, with all other anticommutators vanishing. The fully filled Landau level is a configuration in which all angular momentum eigenstates ψ_k are occupied up to (and including) a maximal angular momentum L: $|\Omega\rangle = a_0^\dagger a_1^\dagger \cdots a_L^\dagger|0\rangle$, where $|0\rangle$ is the Fock vacuum. Since the single-particle angular momentum eigenstates ψ_k are peaked around radii $r_k = \ell\sqrt{k}$, the state $|\Omega\rangle$ consists of a circular droplet of radius approximately given by $\ell\sqrt{L}$ (the boundary is smoothed out by quantum effects). This ground state is *quantum incompressible* in the sense that compressions of the droplet are forbidden by an energy gap. Actually, these would lower the total angular momentum by promoting at least one electron to the next Landau level.

At the quantum level, the boundary density waves of the classical droplet picture become "neutral" particle-hole excitations across the Fermi surface. These are the edge excitations [6] to be included in the low-energy long-distance effective theory, valid in the thermodynamic limit $L \to \infty$. The idea is that to leading nontrivial order in this limit, the filled Landau level can be substituted by an *infinite Dirac sea*, as we now show [7]. We first pick a radius R such that $(R/\ell)^2 = L + \mu, 0 \le \mu < 1$ and we then consider the field operator of the first Landau level restricted to the edge $r = R$, rewritten as follows:

$$\Psi(r = R, \theta) \equiv \left(\frac{2}{\pi\ell^2}\right)^{1/4} e^{i(L+\mu)\theta}\, F_R(\theta) , \qquad (3)$$

where

$$\sqrt{R}\, F_R(\theta) = \sum_{k=-L}^{\infty} \frac{c_k}{\sqrt{2\pi}} e^{i(k-\mu)\theta}\, b_k ,$$
$$c_k^2 = \frac{\sqrt{2\pi}}{(L+k)!} \left(\frac{R}{\ell}\right)^{2L+2k+1} e^{-R^2/\ell^2}\Big|_{R^2=(L+\mu)\ell^2} , \qquad (4)$$

and we introduced the shifted operators $b_k \equiv a_{L+k}$. The coefficients c_k are approximated by $c_k \sim \exp\left[-(k-\mu)^2/2L\right]$ for $L \to \infty$ and provide, therefore, a smooth ultraviolet cutoff $|k| < \sqrt{L}$ for the sum over the angular momentum k. To leading order in $1/L$ we can remove this cutoff thereby obtaining the *effective* relativistic chiral fermion field

$$F(\theta) \equiv \lim_{R\to\infty} \sqrt{R}\, F_R = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{i(k-\mu)\theta}\, b_k \qquad (5)$$

in $1+1$ dimensions (Weyl fermion), with boundary conditions determined by the "chemical potential" μ.

Next we define the $W_{1+\infty}$ generators in the relativistic theory [they can equivalently be obtained by quantization of the $\mathcal{L}_{nm}^{(cl)}$ in the Landau level problem and by the limit (5) [4]]:

$$V_n^j = \int_0^{2\pi} d\theta\, F^\dagger(\theta)\, \ddagger\, e^{-in\theta}\, (i\partial_\theta)^j\, \ddagger\, F(\theta)$$
$$= \sum_{k=-\infty}^{\infty} p(k,n,j;\mu)\, b_{k-n}^\dagger\, b_k , \qquad j \ge 0 , \qquad (6)$$

where $\ddagger\ \ddagger$ denotes an ordering of the first-quantized operators $\exp(-i\theta)$ and $i\partial_\theta$ to be specified below, and such that $V_n^{j\,\dagger} = V_{-n}^j$. The coefficients $p(k,n,j;\mu)$ are jth order polynomials in k whose specific form depends on the choice of ordering. Let us now compute the algebra satisfied by the V_n^i's. Given the standard anticommutation rules of the fermionic operators F and $F^\dagger$, the operator part of $[V_n^i, V_m^j]$ reproduces the corresponding first-quantized commutator. We thus obtain

$$[V_n^i, V_m^j] = (jn - im)\, V_{m+n}^{i+j-1} + q(i,j,m,n) V_{m+n}^{i+j-3} + \cdots + \delta_{n+m,0}\, c(n,i,j) . \qquad (7)$$

The first term on the right hand side (r.h.s.) reproduces the classical algebra (1) by the correspondence $\mathcal{L}_{i-n,i} \to V_n^i$, and identifies (7) as the algebra $W_{1+\infty}$ of "quantum area-preserving diffeomorphisms" [5]. The second and higher operator terms on the r.h.s. arise at the quantum level because the V_n^i are polynomials in ∂_θ. Finally, the c-number terms $c(n,i,j)$ represent the relativistic quantum anomaly. This follows from the renormalization of the charges V_0^i . Since we want to measure charges with respect to the original filled Landau level, we adopt the standard relativistic normal ordering procedure of writing all annihilators to the right of creators. Thus, it can be shown [11] that the coefficients $c(n,i,j)$ depend only on the first-quantized ordering $\ddagger\ \ddagger$ and on μ. In particular, they can be made diagonal, $c(n,i,j) = c(n)\delta^{ij}$, for a specific ordering. Independently of our explicit construction, it has been shown [12] in general that there is a unique central extension for the algebra (7) in the $(1+1)$-dimensional relativistic field theory.

The simplest cases of Eq. (7) are

$$[V_n^0, V_m^0] = n\, \delta_{n+m,0} ,$$
$$[V_n^1, V_m^0] = -m\, V_{n+m}^0 ,$$
$$[V_n^1, V_m^1] = (n-m) V_{n+m}^1 + \frac{c}{12}\left(n^3 - n\right) \delta_{n+m,0} , \qquad (8)$$

with $c = 1$ and where $\mu = 1/2$ has been chosen in order to cancel the anomaly in the second commutator [note that Eqs. (8) are independent of the first-quantized ordering $\ddagger\ \ddagger$]. Equations (8) show that V_n^0 and V_n^1 are the oscillator and conformal modes, respectively, and the central charge is $c = 1$, as expected for a Weyl fermion. The index $i+1$ is the conformal spin of the V_n^i currents and n is the moding.

The quantum incompressibility of the filled Landau level in the limit $L \to \infty$ can now be characterized by the infinite set of conditions

$$V_n^i \,|\Omega\rangle = 0, \qquad \forall n > 0\,, i \geq 0\,. \tag{9}$$

In mathematical terms [3], this equation states that the filled Landau level is a *c=1 highest-weight state* of the $W_{1+\infty}$ algebra with weights $V_0^i|\Omega\rangle = 0$, $\forall i \geq 0$. Moreover, all neutral excitations generated by polynomials of V_n^i $(n<0)$ applied to $|\Omega\rangle$ make up a *unitary irreducible highest weight representation of* $W_{1+\infty}$.

There exist also other excitations, corresponding to quasiholes and quasiparticles in the bulk of the droplet [1]. Actually, due to incompressibility any excitation corresponding to a local density deformation in the bulk is transmitted to the edge, where it is seen as a *charged* excitation. In our algebraic language, it can be shown that these charged excitations, together with their towers of neutral excitations, correspond to further irreducible highest-weight representations [11]. All these highest-weight representations define a $W_{1+\infty}$ conformal field theory, in short, a $W_{1+\infty}$ *theory*. This is the theory of the free Weyl fermion in the case of the filled Landau level discussed so far.

We are thus naturally led to characterize *all two-dimensional incompressible quantum fluids with their edge excitations as* $W_{1+\infty}$ *theories* [actually, analogs of the conditions (9) are known [11,13] for the Laughlin and hierarchical fluids]. This is a powerful classification scheme, given that all highest-weight representations of $W_{1+\infty}$ have been recently obtained by Kac and Radul [8]. Specifically, we are interested in classifying two-dimensional incompressible fluids by their *filling fraction* ν and by the *charge* and *spin (statistics)* quantum numbers of their *quasiparticle excitations.* These are the eigenvalues of the operators $\left(-V_0^0\right)$ and V_0^1 in the highest-weight representation corresponding to a given excitation. Note that the charges of bulk excitations are $-V_0^0$, with opposite sign to their edge counterparts. Moreover, the statistics θ/π, computed from monodromies along the edge, is twice the spin.

The results of Kac and Radul relevant to our construction can be conveniently rephrased in our basis as follows. All unitary, irreducible, highest-weight representations of $W_{1+\infty}$ are completely characterized by a m-dimensional multiplet of "charges," the "vector" $\mathbf{s}$, giving the anomaly c and the highest weights Q and J of $-V_0^0$ and V_0^1, respectively,

$$\begin{aligned}
\mathbf{s} &\equiv \{s^I \in \mathbf{R},\ I=1,\ldots,m\}\,, \quad c=m\,,\ m \in \mathbf{Z}_+\,,\\
Q &= s^1+s^2+\cdots+s^m\,,\\
J &= \frac{1}{2}\left[(s^1)^2+(s^2)^2+\cdots+(s^m)^2\right]\,.
\end{aligned} \tag{10}$$

According to the rules of conformal field theory [3], a $W_{1+\infty}$ theory is defined by a set of these representations which is closed under the "fusion algebra," in physical terms the making of composite excitations. The "fusion rule" for $W_{1+\infty}$ is the addition of charge vectors $\mathbf{s}$, as in the well-known case of the affine subalgebra (8), because the unitary representations of these two algebras are in one-to-one correspondence [11]. Therefore, a consistent set of representations is given by a *lattice* Γ generated by m "elementary" excitations $\mathbf{v}_i$:

$$\Gamma = \left\{\mathbf{s}:\ \mathbf{s} = \sum_{i=1}^{m} n_i \mathbf{v}_i,\ n_i \in \mathbf{Z},\ i=1,\ldots,m\right\}. \tag{11}$$

These results tell us immediately that a level $c=m$ incompressible quantum fluid is a composite with m edges. Indeed, a $c=m$ $W_{1+\infty}$ representation looks like a superposition of m $c=1$ $W_{1+\infty}$ representations, for which the charges can be written as $V_n^0 = \sum_{I=1}^m \left(V_n^0\right)^I$. The crucial point is that the $\left(V_n^0\right)^I$ need not be the appropriate basis describing the *physical charge operators* ρ_0^I on the m edges. These are generically given by

$$\rho_n^I = \sum_{J=1}^{m} \Lambda_{IJ}\,\left(V_n^0\right)^J\,, \qquad \Lambda \in \mathrm{GL}(m,\mathbf{R})\,. \tag{12}$$

The matrix Λ is operationally determined by the chiral couplings of the physical charge densities to independent electromagnetic fields A^I on the edges,

$$\begin{aligned}
H = H_0 + H_{\mathrm{e.m.}} &= \sum_{I=1}^{m}\left(\frac{v^I}{R^I}\left(V_0^1\right)^I + \int_0^{2\pi} d\theta\,\rho^I A^I\right),\\
\rho^I\left(\theta - \frac{v^I}{R^I}t\right) &\equiv \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \rho_n^I\, e^{in(\theta - tv^I/R^I)}\,,\\
A^I &\equiv \left(A_0 + v^I A_\theta\right)|_{r=R^I}\,.
\end{aligned} \tag{13}$$

In Eq. (13), H_0 describes the relativistic chiral dynamics of the edge modes provided by the confining boundary potential [6], which is parametrized phenomenologically by the effective velocities v^I on the m edges at radii R^I. Here we consider only chiral incompressible fluids, for which all v^I have the same sign; the generalization to both chiral and antichiral edges is straightforward.

The *chiral anomaly* generated by the coupling (13) is completely fixed by the current algebra (8) and Eq. (12). It determines the charges created by the electric fields E^I out of the vacuum:

$$\begin{aligned}
\Delta\left(V_0^0\right)^I\Big|_{t=-\infty}^{t=\infty} &\equiv -\sum_J \Lambda_{JI}\int_{-\infty}^{+\infty} dt \int_0^{2\pi}\frac{d\theta}{2\pi}E^J\\
&= -\sum_J \Lambda_{JI} n_J\,,
\end{aligned} \tag{14}$$

where the n_J are integers due to the topological quantization of the electric field in $1+1$ dimensions, and represent the number of *vortices* (quasiparticles) created in the Ith component of the fluid. This spectrum of $\left(V_0^0\right)^I$ must coincide with the spectrum given by the lattice (11), which identifies $\Lambda_{iJ} \equiv (v_i)^J$. We have shown that, given the lattice Γ, the anomaly fixes the definition of the physical charge (12), which in turn implies an "interaction" among the edges with respect to the diagonal basis (10).

The spectrum of physical charges follows directly from (12) and (14),

$$Q \equiv \sum_{I=1}^{m} \rho_0^I = \sum_{i,j=1}^{m} K_{ij}^{-1} n_j , \qquad (15)$$

where we call $K_{ij}^{-1} = \sum_I v_i^I v_j^I$ the *metric* of the lattice. The spin spectrum follows similarly from (10) and (11):

$$J = \frac{\theta}{2\pi} = \frac{1}{2} \sum_{i,j=1}^{m} n_i K_{ij}^{-1} n_j . \qquad (16)$$

Equations (15) and (16) give us the charge, spin, and statistics of the quasiparticle excitation corresponding to any point of the lattice, parametrized by the vorticity components $n_i \in \mathbf{Z}$.

Finally, the filling fraction ν is easily computed by applying a homogeneous tangential electric field $E^I = E$, for all I. The rate of total physical charge created by the anomaly on the edge is $\partial_t(-Q) = -E \sum_{ij} K_{ij}^{-1}$, which corresponds to a radial flux of charge, i.e., to the Hall current [7,10]. This identifies the filling fraction as

$$\nu = \sum_{i,j=1}^{m} K_{ij}^{-1} . \qquad (17)$$

We thus reach the conclusion that all two-dimensional incompressible quantum fluids are classified by an integer level m (the central charge of $W_{1+\infty}$) and a real positive-definite symmetric matrix K^{-1} describing the metric of the representation lattice. Note that our approach cannot predict gaps and excitation energies. Nevertheless, a stability principle based on maximal symmetry of K selects the most prominent fractions observed in experiments [11].

In applying the results (15)–(17) to the quantum Hall effect, we should further require the presence of m excitations with unit charge, fermionic statistics, and integer statistics relative to any other excitation, representing the original electrons in each of the m components of the fluid. This imposes that K has *integer* entries, *odd* on the diagonal [14].

The results (15)–(17) are in agreement with the general hierarchy obtained by Fröhlich, Wen, and Zee [14] by an Abelian Chern-Simons effective field theory. Examples of K matrices for the best known filling fractions have been discussed by these authors; we refer to them for a full discussion of the physical consequences of Eqs. (15)–(17).

While following the same physical approach of describing the universal long-distance properties, the methods of Ref. [14] differ substantially from ours. They started from an appropriate choice of effective field theory, whereas we derived our results from purely algebraic considerations, after having identified the *symmetry principle* governing the phenomenology of the quantum Hall effect. A direct relation between the two approaches can be found *a posteriori*. The $(1+1)$-dimensional m-component chiral boson, compactified on the torus $T^m = \mathbf{R}^m/\Gamma$, gives an explicit realization [11] of all (rational) $W_{1+\infty}$ theories, and it is also the edge degree of freedom of the Abelian Chern-Simons topological theory on a disk [6].

Moreover, our classification shows that the results (15)–(17) are *complete*. We can in fact exclude further possible effective theories: orbifold compactifications of the chiral boson [3] lead to excitations with ill-defined charge V_0^0; non-Abelian Chern-Simons theories possess excitations with non-Abelian statistics which are described by edge theories with noninteger central charge [6]. Both possibilities cannot realize the $W_{1+\infty}$ symmetry.

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