Bethe-Ansatz for the Bloch Electron in Magnetic Field

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(Received 14 September 1993)

We present a new approach to the problem of Bloch electrons in a magnetic field, by making explicit a natural relation between magnetic translations and the quantum group $U_q(sl_2)$. The approach allows us to express the spectrum and the Bloch function as solutions of the Bethe-ansatz equations typical for completely integrable quantum systems.

PACS numbers: 72.15.Gd

A peculiar problem of Bloch electrons in a magnetic field frequently arises in many different physical contexts. Every time it emerges with a new face to describe another physical application [1–5] (i) it resembles some properties of the integer Hall effect [5], (ii) its spectrum has an extremely rich structure of the Cantor set, and exhibits a *multifractal* behavior [4,6,7] (see also [8] for a review), (iii) it describes the localization phenomenon in *quasiperiodic* potential (see, e.g., [8] and references therein), (iv) it has been recently conjectured that the symmetry of the magnetic group may appear dynamically in strongly correlated electronic systems [9,10].

In this paper we show that this problem (sometimes called the Hofstadter problem) and a class of quasiperiodic equations are solvable by the Bethe-ansatz. We made explicit a long time anticipated [11] connection of the group of magnetic translations with the *quantum group* $U_q(sl_2)$ and with quantum integrable systems. The result of the paper is the algebraic Bethe-ansatz equations for the spectrum. Although we do not solve the Bethe-ansatz equations here, we are confident that they provide a basis for analytical study of the multifractal properties of the spectrum.

The Hamiltonian of a particle on a two-dimensional square lattice in a magnetic field is

$$H = \sum_{<n,m>} e^{iA_{\mathbf{n},\mathbf{m}}} c^\dagger_{\mathbf{n}} c_{\mathbf{m}}, \tag{1}$$

$$\prod_{\text{plaquette}} e^{iA_{\mathbf{n},\mathbf{m}}} = e^{i\Phi}, \tag{2}$$

where $\Phi = 2\pi\frac{P}{Q}$ is a flux per plaquette, P and Q are mutually prime integers. In the most conventional Landau gauge $A_x = A_{\mathbf{n},\mathbf{n}+\mathbf{1}_x} = 0, A_y \doteq \Phi n_x$ the Bloch wave function is

$$\psi(\mathbf{n}) = e^{i\mathbf{k}\cdot\mathbf{n}}\psi_{n_x}(\mathbf{k}),\ \psi_n = \psi_{n+Q}, \tag{3}$$

where $n_x \equiv n = 1, ..., Q$ is a coordinate of the magnetic cell. With these substitutions the Schrödinger equation turns into a famous one-dimensional quasiperiodic difference equation ("Harper's" equation):

$$e^{ik_x}\psi_{n+1} + e^{-ik_x}\psi_{n-1} + 2\cos(k_y + n\Phi)\psi_n = E\psi_n. \tag{4}$$

The spectrum of this equation has Q bands and feels the difference between rational and irrational numbers —if the flux is irrational, the spectrum is a singular continuum—uncountable but measure zero set of points (Cantor set).

To ease the reference we first state the main result: It is known that due to the gauge invariance, the energy depends on a single parameter $\lambda = \cos(Qk_x) + \cos(Qk_y)$. We find that the spectrum at $\lambda = 0$ (midband spectrum) is given by the sum of roots z_l,

$$E = iq^Q(q - q^{-1}) \sum_{l=1}^{Q-1} z_l, \tag{5}$$

of the Bethe-ansatz equations for the quantum group $U_q(sl_2)$,

$$\frac{z_l^2 + q}{qz_l^2 + 1} = q^Q \prod_{m=1, m\neq l}^{Q-1} \frac{qz_l - z_m}{z_l - qz_m},\ l = 1, ..., Q-1 \tag{6}$$

with

$$q = e^{\frac{i}{2}\Phi}. \tag{7}$$

The solution of the model with anisotropic hopping [when the coefficient in front of cos in Eq. (4) differs from 2] is also available. It will be published elsewhere.

The quantum group symmetry is more transparent in another gauge $A_x = -\frac{\Phi}{2}(n_x + n_y)$, $A_y = \frac{\Phi}{2}(n_x + n_y + 1)$. In this gauge a discrete coordinate of the Bloch function $\psi(\mathbf{n}) = \exp(i\mathbf{p}\cdot\mathbf{n})\psi_n(\mathbf{p})$, turns to $n = n_x + n_y$. It is defined in two magnetic cells: $n = 1, ..., 2Q$, $p_\pm = (p_x \pm p_y)/2 \in [0, \Phi/2]$. An equivalent form of Harper's equation (4) for ψ_n is

$$2e^{\frac{i}{4}\Phi + ip_+}\cos(\tfrac{1}{2}\Phi n + \tfrac{1}{4}\Phi - p_-)\psi_{n+1} + 2e^{-\frac{i}{4}\Phi - ip_+}\cos(\tfrac{1}{2}\Phi n - \tfrac{1}{4}\Phi - p_-)\psi_{n-1} = E\psi_n. \tag{8}$$

0031-9007/94/72(12)/1890(4)$06.00

In the new gauge the midband $\lambda = 0$ corresponds to the point $\mathbf{p} = (\frac{1}{2}\pi, \frac{1}{2}\pi)$. [The doubling of period in comparison with original Harper's equation is artificial: using a simple transformation (multiplying ψ_n by $e^{-i\Phi n^2/4}$) one comes to an equation with coefficients of period Q.] The advantage of this gauge is that the wave function turns into the polynomial with roots z_m,

$$\Psi(z) = \prod_{m=1}^{Q-1} (z - z_m), \tag{9}$$

at the points $z = q^{2l}$,

$$\psi_n = \Psi(q^n). \tag{10}$$

The wave function of a particle in a magnetic field forms a representation of the *group of magnetic translations* [1]: generators of magnetic translations

$$T_\mu(\mathbf{i}) = \exp(iA_{\mathbf{i},\mathbf{i}+\mu})\,|\,\mathbf{i}\rangle\langle\mathbf{i}+\boldsymbol{\mu}\,| \tag{11}$$

form the algebra

$$T_\mu = T_{-\mu}^{-1},\ \ T_\mathbf{n}T_\mathbf{m} = q^{-\mathbf{n}\times\mathbf{m}}T_{\mathbf{n}+\mathbf{m}}, \\ T_yT_x = q^2T_xT_y,\ \ T_yT_{-x} = q^{-2}T_{-x}T_y\cdots, \tag{12}$$

with q given by Eq. (7). The Hamiltonian (1) can be expressed

$$H = T_x + T_{-x} + T_y + T_{-y}. \tag{13}$$

The algebra $U_q(sl_2)$ (a q deformation of the universal enveloping of the sl_2) is generated by the elements A, B, C, D, with the commutation relations [12–16]

$$AB = qBA,\ \ BD = qDB, \\ DC = qCD,\ \ CA = qAC, \\ AD = 1,\ \ [B,C] = \frac{A^2 - D^2}{q - q^{-1}}. \tag{14}$$

The center of this algebra is a q analog of the Casimir operator

$$c = \left(\frac{q^{-\frac{1}{2}}A - q^{\frac{1}{2}}D}{q - q^{-1}}\right)^2 + BC. \tag{15}$$

In the limit $q \to 1 + \frac{i}{2}\Phi$ (so-called classical limit), the quantum group turns to the sl_2 algebra: $(A - D)/(q - q^{-1}) \to S_3$, $B \to S_+$, $C \to S_-$, $c \to \mathbf{S}^2 + 1/4$.

The commutation relations (14) are simply another way to write the Yang-Baxter equation

$$R_{a_1a_2}^{b_1b_2}(u/v)L_{b_1c_1}(u)L_{b_2c_2}(v) = L_{a_1b_1}(u)L_{a_2b_2}(v)R_{b_1b_2}^{c_1c_2}(u/v), \tag{16}$$

where generators A, B, C, D are matrix elements of the L operator

$$L(u) = \begin{bmatrix} \frac{uA-u^{-1}D}{q-q^{-1}} & u^{-1}C \\ uB & \frac{uD-u^{-1}A}{q-q^{-1}} \end{bmatrix}. \tag{17}$$

Here u is a spectral parameter and R matrix is the L operator in the spin $\frac{1}{2}$ representation. It is given by the same matrix (17) with elements: $A = q^{\frac{1}{2}\sigma_3}$, $D = q^{-\frac{1}{2}\sigma_3}$, $B = \sigma_+$, $C = \sigma_-$, where $\boldsymbol{\sigma}$ are the Pauli matrices.

Finite-dimensional representations (except some representations of dimension Q) of the $U_q(sl_2)$ can be expressed in the weight basis, where A and D are diagonal matrices: $A = \operatorname{diag}(q^j, ..., q^{-j})$. An integer or half integer j is the spin of the representation, and $2j + 1$ is its dimension. The value of the Casimir operator (15) in this representation is the q analog of $(j + 1/2)^2$

$$c = \left(\frac{q^{j+1/2} - q^{-j-1/2}}{q - q^{-1}}\right)^2 = [j + 1/2]_q^2. \tag{18}$$

Representations can be realized by polynomials $\Psi(z)$ of the degree $2j$:

$$A\Psi(z) = q^{-j}\Psi(qz),\ \ D\Psi(z) = q^j\Psi(q^{-1}z), \\ B\Psi(z) = z(q - q^{-1})^{-1}\left[q^{2j}\Psi(q^{-1}z) - q^{-2j}\Psi(qz)\right], \\ C\Psi(z) = -z^{-1}(q - q^{-1})^{-1}\left[\Psi(q^{-1}z) - \Psi(qz)\right]. \tag{19}$$

This again is the q analog of the representation of the sl_2 algebra by a differential operator:

$$S_3 = z\frac{d}{dz} - j,\ \ S_+ = z\left(2j - z\frac{d}{dz}\right),\ \ S_- = \frac{d}{dz}. \tag{20}$$

The dimension of our physical space of states is $2j+1 = Q$. This is a very special dimension when $q^{2j+1} = \mp 1$ for P odd (even). The Casimir operator (18) in this case is

$$c = -4(q - q^{-1})^{-2},\ \text{ for } P \text{ odd}, \\ c = 0,\ \text{ for } P \text{ even}. \tag{21}$$

In this special case representation of the quantum group can be naturally expressed in terms of magnetic translations

$$T_{-x} + T_{-y} = \pm i(q - q^{-1})B, \\ T_x + T_y = i(q - q^{-1})C, \\ T_{-y}T_x = \pm q^{-1}A^2,\ \ T_{-x}T_y = \pm qD^2, \tag{22}$$

where the upper sign corresponds to an odd P and the lower to an even P [representation of $U_q(sl_2)$ in terms of different but related Weyl basis can be found in Refs. [17–20]]. It is straightforward to check that this representation obeys commutation relations (12) and (14) and gives correct values (21) of the Casimir operator (15).

The Hamiltonian (1) and (13) now can be expressed in terms of the quantum group generators

$$H = i(q - q^{-1})(C \pm B), \tag{23}$$

whereas the Schrödinger equation becomes a difference functional equation

$$i(z^{-1} + qz)\Psi(qz) - i(z^{-1} + q^{-1}z)\Psi(q^{-1}z) = E\Psi(z). \tag{24}$$

The original Harper's discrete equation in the form (8) at the point $\mathbf{p} = (\pi/2, \pi/2)$ can be obtained from the functional equation (24), by setting $z = q^l$ and $\psi_l = \Psi(q^l)$. The advantage to use the extention of ψ_l to a complex plane z is that the representation theory of the quantum group guarantees that in a proper gauge the extended wave function would be the polynomial (9).

In addition to representation (19) having the highest and the lowest weight, in the special dimension $q^{2j+1} = \pm 1$ there is a parametric family of representations having in general no highest or lowest weights [18,19]. The parameter describes the anisotropy of the hopping amplitude in the Hamiltonian (1) and momentum $\mathbf{k}$ differs from the band's center. The Bethe-ansatz solution of the anisotropic problem is also possible but requires much heavier mathematical technique.

Among various methods of the theory of quantum integrable systems the *functional Bethe-ansatz* [21] seems to be the most direct way to diagonalize the Hamiltonian (23). It is simple. We know that the solution of Eq. (24) is the polynomial

$$\Psi(z) = \prod_{m=1}^{Q-1} (z - z_m). \tag{25}$$

Let us substitute it in Eq. (24) and divide both sides by $\Psi(z)$. We obtain

$$i(z^{-1} + qz) \prod_{m=1, m\neq l}^{Q-1} \frac{qz - z_m}{z - z_m} - i(z^{-1} + q^{-1}z) \prod_{m=1, m\neq l}^{Q-1} \frac{q^{-1}z - z_m}{z - z_m} = E. \tag{26}$$

The left-hand side (l.h.s.) of this equation is a meromorphic function, whereas the right-hand side (r.h.s.) is a constant. To make them equal we must null all residues of the l.h.s. They appear $z = 0$, $z = \infty$, and at $z = z_m$. The residue at $z = 0$ vanishes automatically.

The residue at $z = \infty$ is $-iq^Q + iq^{-Q}$. Its null determines the degree of the polynomial.

Comparing the coefficients of z^{Q-1} on both sides of Eq. (24), we obtain the energy given by Eq. (5).

Finally, annihilation of poles at $z = z_m$ gives the Bethe-ansatz equations (6) for roots of the polynomial (9). We write them here in a more conventional form by setting $z_l = \exp(2\varphi_l)$,

$$\frac{\cosh(2\varphi_l - i\frac{\Phi}{4})}{\cosh(2\varphi_l + i\frac{\Phi}{4})} = \mp \prod_{m=1, m\neq l}^{Q-1} \frac{\sinh(\varphi_l - \varphi_m + i\frac{\Phi}{4})}{\sinh(\varphi_l - \varphi_m - i\frac{\Phi}{4})}. \tag{27}$$

Harper's equation is just a representative (perhaps the most interesting one) of the class of solvable discrete equations of the second order

$$a_n\psi_{n+1} + b_n\psi_{n-1} - c_n\psi_n = E\psi_n \tag{28}$$

where periodic coefficients obey a certain integrability condition [22]. All of them may be obtained from a general quadratic form of quantum group generators. Some of these equations have direct physical interpretation. Their classical version ($q \to 1$) would be differential equations generated by quadratic forms of sl_2 generators (20). These differential equations are known in the literature as "quasi-exactly-soluable" problems of quantum mechanics [23,24] and in particular includes differential equations for classical orthogonal polynomials. In turn some of the discrete equations (28) generate a q analog of orthogonal polynomials [25]. For example, the zero energy solution of Harper's equations (8) and (24) is the q Legendre polynomial of the degree $\frac{Q-1}{2}$. Its explicit form is known:

$$\Psi^{(E=0)}(iz) = z^{(Q-1)/2} = \sum_{m=0}^{n} \frac{(q;q)_n (q^{1/2};q)_m (q^{1/2};q)_{n-m}}{(q;q)_m (q;q)_{n-m}} z^{2m-n},$$

where $(a;q)_n = \prod_{l=0}^{n-1}(1 - aq^l)$ is the standard notation.

The most interesting feature of Harper's equation is the multifractality of the spectrum at $P, Q \to \infty$, i.e., when flux $\Phi/2\pi$ is irrational. The spectrum is complex but not chaotic. Quite the opposite, it is determined by the quantum integrability. Moreover, we have shown that the distribution of the spectrum is equivalent to the distribution of zeros of the q analog of hypergeometrical functions.

As we already mentioned the Bethe-ansatz solution is available (but not presented here) for an arbitrary strength of the potential of Harper's equation. It provides the basis to study Anderson localization-delocalization transition of electrons in quasiperiodic potential.

To the best of our knowledge even elementary questions regarding the Hofstadter problem at irrational flux remained unclear. For example, the low temperature thermodynamics, dispersion of excitations, conductivity, etc., are not known. In all previous examples of quantum integrable models all these quantities have been found by solving Bethe-ansatz equations. Moreover, multifractal properties of the spectrum have already appeared in the solution of the XXZ magnetic chain [26] and the Sine-Gordon model [27]. However, this aspect of integrability has never been developed. The Bethe-ansatz solution does not give an advantage at finite P and Q. Just contrary, up to $Q \sim 6000$ direct diagonalization of the

Hofstadter Hamiltonian is more effective. However, as usual the Bethe-ansatz solution becomes a powerful tool at $Q \to \infty$. The strategy of solving the Bethe equations is well known: at large Q the roots z_l form dense groups ("strings") and can be described by their distributions. The algebraic equations are then replaced by the system of integral equations for the distribution functions of strings. This program is in progress.

We would like to thank A. Abanov, E. Floratos, P.G.O. Freund, J.-L. Gervais, A. Gorsky, L. Kadanoff, A. Kirillov, L. Levitov, B. Shraiman, and V. Pasquier for interesting discussions. A.Z. is grateful to the Mathematical Disciplines Center of the University of Chicago for hospitality and support. P.W. acknowledges the hospitality of Weizmann Institute of Science and Laboratoire de Physique Theorique de l'Ecole Normale Supérieure where this work was completed. This work was supported in part by NSF under the Research Grants No. 27STC-9120000 and No. NSF-DMR 88-19860.

Note added.—Recently L.D. Faddeev and R.M. Kashaev [28] accomplished an impressive generalization of the Bethe-ansatz solution of Harper's equation for anisotropic hopping amplitudes t_x and t_y and arbitrary momentum **k**. In particular their result states that for the midband spectrum the anisotropy changes the l.h.s. of the Bethe-ansatz equations (6) by $\frac{(t_x z_l + iq^{1/2})(t_y z_l + iq^{1/2})}{(q^{1/2} t_x z_l + i)(q^{1/2} t_y z_l + i)}$ whether the r.h.s. remains the same.

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