Supersymmetry of the (2+1)-Dimensional Black Holes

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The supersymmetry properties of the asymptotically anti–de Sitter (adS) black holes of Einstein theory in 2+1 dimensions are investigated. It is shown that (i) the zero-mass black hole has two exact supersymmetries; (ii) extreme $lM = |J|$ black holes with $M \neq 0$ have only one; and (iii) generic black holes do not have any. It is also argued that the zero-mass hole is the ground state of (1,1) adS supergravity with periodic ("Ramond") boundary conditions on the spinor fields.

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Among the black hole solutions of (2+1) Einstein theory discovered recently [1], the one with zero mass and zero angular momentum stands apart. (i) It is the solution with smallest mass. (ii) It has zero temperature. (iii) It has zero entropy. We show in this Letter that it enjoys also remarkable supersymmetry properties [2–10]. Namely, it is the black hole solution with the maximum number of exact supersymmetries. We shall first establish the result and shall then discuss its implications [11].

The 2+1 black hole metric is given by [1]

$$\begin{aligned} ds^2 &= -N^2dt^2 + N^{-2}dt^2 + r^2(N^\varphi dt + d\varphi)^2, \\ N^2 &= (r/l)^2 - M + (J/2r)^2, \\ N^\varphi &= -J/2r^2, \end{aligned} \tag{1}$$

where M and J are, respectively, the mass and angular momentum of the hole, and where $-l^2$ is the cosmological constant. It can be obtained by making appropriate identifications of the anti–de Sitter (adS) metric [12], which corresponds to (1) with $M = -1$ and $J = 0$,

$$ds^2_{\rm adS} = -[(r/l)^2+1]dt^2 + [(r/l)^2+1]^{-1}dr^2 + r^2d\varphi^2. \tag{2}$$

The metric (1) with $M \neq -1$ has only two Killing vectors [12]. If regarded as a solution of the equations of motion of adS supergravity with zero gravitini, it may possess, in addition, exact supersymmetries. Exact supersymmetries are by definition supersymmetry transformations leaving the metric (1) (with zero gravitini) invariant. The spinor parameters of these transformations solve the "Killing spinor equation"

$$D_\lambda \psi = \frac{\epsilon}{2l}\gamma_\lambda \psi, \tag{3}$$

where $\epsilon = 1$ or -1 depending on the representation of the γ matrices.

As is well known, there are two inequivalent two-dimensional irreducible representations of the γ matrices in three spacetime dimensions. One may be taken to be $\gamma^{(0)} = i\sigma^2$, $\gamma^{(1)} = \sigma^1$, and $\gamma^{(2)} = \sigma^3$, where the σ^k are the Pauli matrices. The other is given by $\gamma'^{(\lambda)} = -\gamma^{(\lambda)}$. We shall consider here the simplest supergravity model with negative cosmological constant involving both representations, namely, $(1,1)$ adS supergravity [13].

The anti–de Sitter metric (2) possesses four Killing spinors, two for each inequivalent representation of the γ matrices. In the radial tetrad frame

$$\begin{aligned} h_{(0)} &= -[(r/l)^2+1]^{\frac12}dt, \\ h_{(1)} &= [(r/l)^2+1]^{-\frac12}dr, \\ h_{(2)} &= r d\varphi, \end{aligned} \tag{4}$$

the Killing spinors are given by

$$\psi = \left[\left(\frac{N_{\rm adS}+1}{2}\right)^{\frac12} + \epsilon\left(\frac{N_{\rm adS}-1}{2}\right)^{\frac12}\gamma^{(1)}\right] \times \left(\cos\frac12(\varphi+\epsilon t/l) - \sin\frac12(\varphi+\epsilon t/l)\gamma^{(0)}\right)A, \tag{5}$$

where A is a constant spinor.

Since the black hole metric can be obtained from (2) by making appropriate identifications, it possesses locally as many Killing spinors as anti–de Sitter space. However, only a subset of these Killing spinors are, in general, compatible with the identifications, i.e., invariant under the transformations of the discrete group used in the identifications. So, whereas all the local integrability conditions for the Killing equations (3) are fulfilled [14], there may be no Killing spinor at all because of global reasons.

In order to discuss which Killing spinors are compatible with the identifications, let us make a choice of coordinates in which the identifications take a simple form. As shown in [12] , anti–de Sitter space can be rewritten as

$$ds^2_{\rm adS} = -[(R/l)^2-1]dT^2 + [(R/l)^2-1]^{-1}dR^2 + R^2d\Phi^2 \tag{6}$$

in new coordinates (T, R, Φ) where Φ is not an angle but runs over the entire real line. The Killing spinors are, in the frame

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$$\begin{aligned} h_{(0)} &= -[(R/l)^2 - 1]^{\frac{1}{2}} dT, \\ h_{(1)} &= [(R/l)^2 - 1]^{-\frac{1}{2}} dR, \\ h_{(2)} &= R\, d\Phi \end{aligned} \tag{7}$$

given by

$$\psi = \frac{1}{\sqrt{2}}\{[(R/l)+1]^{\frac{1}{2}} + \epsilon[(R/l)-1]^{\frac{1}{2}}\gamma^{(1)}\} \times [\cosh\tfrac{1}{2}(\Phi + \epsilon T/l) + \sinh\tfrac{1}{2}(\Phi + \epsilon T/l)\gamma^{(2)}]A. \tag{8}$$

The identifications appropriate to a nonextreme black hole with angular momentum J and mass M ($|J| < Ml$) have been shown in [12] to be

$$(T, \Phi) \sim (T + J, \Phi + M), \quad |J| < Ml. \tag{9}$$

Since the Killing spinors are not invariant (even up to a sign) under these identifications, they are not well defined in the quotient space. Therefore, a generic black hole has no Killing spinor.

Let us now consider the extreme case $|J| = Ml$. The identifications appropriate to that case are more complicated to describe in the coordinate system where (6) holds [16]. For that reason, we shall directly proceed to the explicit integration of the Killing spinor equations in the metric (1), where φ is an angle. For definiteness, we treat the case $J = Ml$. The case $J = -Ml$ is treated similarly. One may take as local Lorentz frame

$$\begin{aligned} h_{(0)} &= -N dt, \\ h_{(1)} &= N^{-1} dr, \\ h_{(2)} &= -\frac{Ml}{2r} dt + r\, d\varphi. \end{aligned} \tag{10}$$

One finds that the Killing spinors solutions of (3) with $\epsilon = 1$ are given by

$$\psi = \frac{1}{2}[(U^{\frac{1}{2}} + U^{-\frac{1}{2}}) + (U^{\frac{1}{2}} - U^{-\frac{1}{2}})\gamma^{(1)}] \times \left[1 + \frac{1}{2}(\gamma^{(2)} - \gamma^{(0)})(\varphi + t/l)\right] A, \tag{11}$$

where A is a constant spinor and U is given by

$$U = l/r[(r/l)^2 - M/2]. \tag{12}$$

The Killing spinors for the other representation of the γ matrices are

$$\begin{aligned} \psi = {} & \frac{1}{2}(r/l)^{\frac{1}{2}}\{\alpha\cosh[\sqrt{M/2}(t/l - \varphi)] \\ & + \beta\sinh[\sqrt{M/2}(t/l - \varphi)]\}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ & + \frac{1}{2}(r/l)^{-\frac{1}{2}}\sqrt{M/2}\{\alpha\sinh[\sqrt{M/2}(t/l - \varphi)] \\ & + \beta\cosh[\sqrt{M/2}(t/l - \varphi)]\}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \end{aligned} \tag{13}$$

where we recall that $\gamma^{(1)} = \sigma^{(1)}$, and where α and β are constants. The Killing spinor (11) is compatible with the periodicity of φ if and only if the linearly growing term in φ disappears, i.e., if and only if A is an eigenstate of $\gamma^{(1)}$ with eigenvalue $+1$. In that case, (11) does not depend on φ and is thus manifestly periodic. The Killing spinor (13) is never periodic or antiperiodic. There is thus only one Killing spinor for the extreme black hole with nonvanishing mass.

In the limit $M \to 0$, one gets from each sign of J a φ-independent Killing spinor. These read explicitly

$$\psi_1 = \frac{1}{2}(r/l)^{\frac{1}{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{14}$$

and

$$\psi_2 = \frac{1}{2}(r/l)^{\frac{1}{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{15}$$

The zero-mass state thus has two exact supersymmetries.

The Killing spinors of the extreme black hole solutions have the same asymptotic growing in r as the Killing spinors of anti–de Sitter space. However, they are periodic in φ, while those of anti–de Sitter space are antiperiodic.

This feature has interesting implications. It has been established in [17] that a negative cosmological constant allows for rich asymptotics. Namely, there exist boundary conditions on the gravitational variables such that the asymptotic symmetry algebra of $(2+1)$ gravity with a negative cosmological constant is the conformal algebra in two dimensions, i.e., twice the Virasoro algebra. The mass and angular momentum are respectively given by $M = l^{-1}(K_0 + L_0)$ and $J = K_0 - L_0$, where K_n and L_n are the right and left Virasoro generators. These boundary conditions include the black hole solutions of [1].

Now, in a spacetime with the black hole topology $R^2 \times S^1$ [12], one can consider spinor fields that are either periodic or antiperiodic in φ in the above radial triad frames. These different behaviors define inequivalent spinor structures and lead to different asymptotic superalgebras for (3+1) supergravity with a negative cosmological constant. The periodic case yields the Ramond graded extension of the Virasoro algebra and will be referred to as the "Ramond sector" for that reason. The antiperiodic case yields the Neveu-Schwarz extension and will be called the "Neveu-Schwarz sector."

Just as in $3+1$ dimensions [18–20], the asymptotic supersymmetry algebra implies bounds for the generators K_0 and L_0. The stronger ones are

$$K_0 = G_{1/2}G_{-1/2} + G_{-1/2}G_{1/2} - \frac{1}{2} \geq -\frac{1}{2}, \tag{16}$$

$$L_0 = \bar{G}_{1/2}\bar{G}_{-1/2} + \bar{G}_{-1/2}\bar{G}_{1/2} - \frac{1}{2} \geq -\frac{1}{2} \tag{17}$$

for the Neveu-Schwarz case, and

$$K_0 = F_0^2 \geq 0, \tag{18}$$

$$L_0 = \bar{F}_0^2 \geq 0 \tag{19}$$

for the Ramond case. The G_k and F_k are the asymptotic right supersymmetry generators, while the $\bar{G}_k$ and $\bar{F}_k$ are the asymptotic left supersymmetry generators.

The exact supersymmetries of anti–de Sitter space belong to the Neveu-Schwarz sector and are generated by the right and left supersymmetry charges with "frequency" 1/2 and $-1/2$ (the Killing spinors have that dependence on φ). Hence, anti–de Sitter space is annihilated by $G_{1/2}$ and $\bar{G}_{1/2}$ and saturates the bound for the Neveu-Schwarz case, in agreement with $M = -1$. Similarly, the zero-mass hole is invariant under the two zero-mode supersymmetries generated by F_0 and $\bar{F}_0$ and saturates the bounds (18) and (19) of the Ramond case, leading to $M = 0$. Accordingly, the zero-mass hole appears as the ground state of the Ramond sector. The extreme black holes $lM = |J|$ with $M \neq 0$ saturate only one of the bounds (18) or (19).

In the above analysis, we have set the electric charge equal to zero. The reason why we did not consider charged black holes is that these appear to possess somewhat unphysical properties in 2+1 dimensions. (i) They fail to fullfill the falloff conditions given in [17] for asymptotically anti–de Sitter spaces. (ii) The energy M is not bounded from either below or above when the charge is different from zero. Indeed given an arbitrarily negative mass, the solution given in [1] possesses an event horizon hiding the singularity for Q big enough. The unboundedness of the energy renders the solutions unstable and should imply the absence of asymptotic Killing spinors *a fortiori* of exact Killing spinors.

Once one imposes the asymptotic behavior of [17], one must take $Q = 0$. This forces the electromagnetic field to vanish and makes the vector potential locally pure gauge. The vector potential is not necessarily globally pure gauge, however, since the fundamental group of black hole solutions is nontrivial. Because of the presence of noncontractible loops, the black hole spacetimes can support nonzero holonomies of A_λ, given by $A_t = 0$, $A_r = 0$, $A_\varphi =$ const. This is somewhat reminiscent of 3+1 black holes with axionic charge [21]. It is no accident, since in 2+1 dimensions, the electromagnetic field (and not the Kalb-Ramond field) is dual to the axion field.

In four-dimensional Einstein-Maxwell theory with zero cosmological constant, the only black holes with exact supersymmetries are the extreme Reissner-Nordstrom black holes. These have the further remarkable property that one can construct static, extreme multi-black-hole solutions [22,23], in which the Coulomb potential exactly balances the gravitational attraction. Both properties have sometimes been related, so that it is natural to ask whether one can also construct static multi-black-hole solutions in 2+1 dimensions. It turns out that this is not the case. As we shall show in detail in a separate publication where the general static solution of 2+1 Einstein theory with negative cosmological constant will be constructed, there is no static, supersymmetric, pure multi-black-hole metric without additional (undesirable) naked branch point singularities.

To conclude, we have shown in this Letter that the zero-mass state enjoys remarkable supersymmetry properties. These indicate that the zero-mass state is the ground state of the Ramond sector of (1,1) adS supergravity. Furthermore, exact supersymmetry is associated with the precise bounds guaranteeing the absence of naked singularity (cosmic censorship), as in 3+1 dimensions [9]. The extreme bound $lM = |J|$ yields one supersymmetry. The bound $M = 0$ yields a second supersymmetry. A detailed presentation of this work, covering the extended (p, q) adS supergravity models, will be reported elsewhere.

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