From Feynman's Wave Function to the Effective Theory of Vortex Dynamics

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We calculate the overlap between two many-body wave functions for a superfluid film containing a vortex at shifted positions. Comparing the results to phenomenological theories, which treat vortices as point particles, we find that the results are consistent if the point-particle vortices are considered as under the action of the Magnus force and in weak interaction with sound waves of the superfluid. We are then able to resolve the disagreement concerning the effective mass of vortices, showing it is finite.

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Vortices play an important role in the understanding of both static and dynamical properties of a superfluid [1]. They determine the Kosterlitz-Thouless phase transition [2], and provide a mechanism for the mutual friction between the superfluid and the normal fluid [3]. As a result of advances in experimental techniques, there are many studies of problems related to vortex dynamics, such as the quantum nucleation of vortex rings induced by moving ions [4] and quantum phase slippage near a submicron orifice [5]. In two dimensions, the theoretical framework for understanding these dynamical phenomena is based on an effective point-particle formulation of vortex dynamics, and has been very successful [1]. Naturally, physical quantities in the phenomenological theory, such as the vortex mass, the Magnus force, and the friction should be derived from a microscopic theory. However, the current understanding of these quantities is in a confused state: There is no clear calculation of the coupling of the vortex to the low lying excitations responsible for the friction, and the theoretical estimates of the vortex mass range from zero [6], to finite [7], to infinite [8]. There is also a suspicion that an effective mass may not be meaningfully defined for a vortex after all [9].

The purpose of the present paper is to present a conceptually straightforward calculation to give clear constraints on these quantities. We invoke a microscopic description of the vortex by writing a Feynman many-body wave function for a superfluid film containing a vortex [10]. We calculate the overlap integral between such a state and that with the vortex shifted a distance away, and find how it behaves as a function of the distance. We also calculate the same quantity within the phenomenological point-vortex theory. Comparing the two, we conclude that the effective mass of the vortex cannot be infinite, and that the coupling of a vortex with low lying excitations must be sufficiently weak. At the end of the paper, we will discuss the generality of our approach and its application to other systems.

Let us start with the phenomenological theory of vortex dynamics in a two dimensional superfluid film. A vortex is regarded as a point particle moving under the influence of the Magnus force $h\rho_0\hat{\mathbf{z}}\times\mathbf{v}$, where h is the Planck constant, ρ_0 is the 2D superfluid number density, $\hat{\mathbf{z}}$ is the unit vector normal to the film, and $\mathbf{v}$ is the velocity of the vortex. Its effective Hamiltonian may be written as

$$H_v=\frac{1}{2m_v}[-i\hbar\nabla-q\mathbf{A}(\mathbf{r})]^2, \quad (1)$$

where m_v is an effective inertial mass of the vortex, the vector potential $\mathbf{A}$ in the symmetric gauge for the Magnus force is $(-y,x)h\rho_0/2$, $\mathbf{r}$ is the vortex coordinate, and $q=\pm 1$ is the vorticity of the vortex. Equation (1) can be understood by drawing an analogy with the case of a two dimensional electron moving in a magnetic field, with q interpreted as the vortex "charge" [11].

This simple phenomenology is unfortunately not adequate if one wishes to compare with a more microscopic theory. One must also include interactions with low lying excitations such as various sound waves of the superfluid, which may be realized by the following model:

$$H_i=q\sum_{\mathbf{k}}M(k)e^{i\mathbf{k}\cdot\mathbf{r}}(a_{\mathbf{k}}+a^{\dagger}_{-\mathbf{k}}), \quad (2)$$

where $\mathbf{k}$ is the wave vector of a low lying excitation with the corresponding creation (annihilation) operator $a^{\dagger}_{\mathbf{k}}$ ($a_{\mathbf{k}}$). An index labeling different kinds of excitations is omitted for notational simplicity. Coupling of this form conserves the total momentum of the system, as is necessary for a translationally invariant system. The Hamiltonian for the low lying excitations is $H_e=\sum_{\mathbf{k}}\hbar\omega_k a^{\dagger}_{\mathbf{k}}a_{\mathbf{k}}$. Therefore the total Hamiltonian of the system, a vortex and the low lying excitations, is $H=H_v+H_i+H_e$. We will focus our attention on the overlap integral between different vortex states, in which the vortex mass m_v, the Magnus force, and the coupling between the vortex and the low lying excitations should be involved.

In the absence of the coupling to the low lying excitations, the overlap integral between two coherent states centered at $\mathbf{r}_0'$ and $\mathbf{r}_0$ can be calculated as [11]

$$O(\mathbf{r}_0,\mathbf{d})=\langle\mathbf{r}_0'|\mathbf{r}_0\rangle=\exp\left[-\frac{|\mathbf{d}|^2}{4l_m^2}+\frac{i}{2l_m^2}\hat{\mathbf{z}}\cdot(\mathbf{d}\times\mathbf{r}_0)\right], \quad (3)$$

where $l_m=(2\pi\rho_0)^{-1/2}$ is the mean spacing between the atoms in the superfluid, and $\mathbf{d}=\mathbf{r}_0-\mathbf{r}_0'$. The coherent state has the form of (4) below, with l and l' replaced by l_m, and $|\psi_e\rangle$ by the vacuum of the low lying excitations. The above overlap integral contains a phase factor, derived from the Berry phase (or Aharonov-Bohm phase in this context) of the coherent state. It also contains a Gaussian decay factor, reflecting the localization of the coherent state. Both factors are characterized by l_m, and are independent of the vortex mass m_v.

In the presence of interactions with the low lying excitations, the total overlap integral will change in two ways by the polaron effect [12]: (a) The vortex can induce polarization of the excitations, and the overlap between the polarized excitations of one coherent state of the vortex and those of a shifted coherent state can contribute to the reduction of the total overlap integral. (b) The polarized excitations tend to localize the vortex, squeezing the coherent state to a smaller size than l_m. These effects will clearly depend on the interaction strength, and will also involve the vortex mass. Now a coherent state of the vortex centered at $\mathbf{r}_0$ may be approximated by the following variational wave function:

$$|\mathbf{r}_0\rangle=\frac{1}{\sqrt{2\pi l^2}}\exp\left[-\frac{|\mathbf{r}-\mathbf{r}_0|^2}{4l^2}+\frac{i\hat{\mathbf{z}}\cdot\mathbf{r}_0\times\mathbf{r}}{2l'^2}\right]\times|\psi_e\rangle, \qquad (4)$$

where l and l' are two variational parameters, and $|\psi_e\rangle$ is a wave function of the excitations only. With the above ansatz, the total energy of the system is evaluated as

$$E=\frac{\hbar^2}{4m_v l_m^2}\left[\frac{l^2}{l_m^2}+\frac{l_m^2}{l^2}+\frac{\mathbf{r}_0^2}{2}\left(\frac{1}{l_m}-\frac{l_m}{l'^2}\right)^2\right]+\langle\psi_e|(H_e+\bar{H}_i)|\psi_e\rangle, \qquad (5)$$

where $\bar{H}_i=q\sum_{\mathbf{k}}M(k)e^{-k^2l^2/2}e^{i\mathbf{k}\cdot\mathbf{r}_0}(a_{\mathbf{k}}+a^\dagger_{-\mathbf{k}})$. First, the energy is minimized by taking $|\psi_e\rangle$ as the ground state of $H_e+\bar{H}_i$, namely,

$$|\psi_e\rangle=\exp\left[q\sum_{\mathbf{k}}\frac{M(k)e^{-k^2l^2/2}}{\hbar\omega_k}e^{i\mathbf{k}\cdot\mathbf{r}_0}(a_{\mathbf{k}}-a^\dagger_{-\mathbf{k}})\right]|0\rangle, \qquad (6)$$

where $|0\rangle$ is the vacuum of the excitations. The energy of the system then becomes

$$E=\frac{\hbar^2}{4m_v l_m^2}\left[\frac{l^2}{l_m^2}+\frac{l_m^2}{l^2}+\frac{\mathbf{r}_0^2}{2}\left(\frac{1}{l_m}-\frac{l_m}{l'^2}\right)^2\right]-\sum_{\mathbf{k}}\frac{|M(k)|^2e^{-k^2l^2}}{\hbar\omega_k}. \qquad (7)$$

Obviously $l'=l_m$ minimizes (7). The energy is further minimized with respect to l if

$$l^{-4}=l_m^{-4}+\frac{4m_v}{\hbar}\int_0^\infty d\omega\frac{J(\omega)}{\omega}e^{-k_\omega^2l^2}, \qquad (8)$$

where the spectral function $J(\omega)$ is defined as

$$J(\omega)=\sum_{\mathbf{k}}\frac{|M(k)|^2k^2}{\hbar^2}\delta(\omega_k-\omega). \qquad (9)$$

Having the variational parameters l and l' determined, the overlap integral $O(\mathbf{r}_0,\mathbf{d})$ is then found as

$$O(\mathbf{r}_0,\mathbf{d})=\langle\mathbf{r}_0'|\mathbf{r}_0\rangle=\exp\left[-\frac{|\mathbf{d}|^2}{4l_d^2}+\frac{1}{2l_m^2}(i\hat{\mathbf{z}}\cdot\mathbf{r}_0\times\mathbf{d})\right] \qquad (10)$$

for a sufficiently small distance $|\mathbf{d}|$. Here the decay length l_d in (10) is

$$\frac{1}{l_d^2}=\frac{1}{2l^2}+\frac{l^2}{2l_m^4}+\int_0^\infty d\omega\frac{J(\omega)}{\omega^2}e^{-k_\omega^2l^2}. \qquad (11)$$

The above results have several interesting features. First, the length in the Berry phase term is not renormalized by the interactions. In fact, the same result is reached even if we assume a general phase factor in the ansatz (4). The result has nicely demonstrated the robustness of the Berry phase against the details. Second, by (8) the localization length l is smaller than l_m. The effective mass m_v enters in the equation, because it determines the Landau level spacing, which in turn tells how hard it is to mix with the higher Landau levels in order to shrink l. Third, the last term of (11) comes from the overlap of the polarized excitations. Finally, when we consider the contribution from the fluctuating vector potential in (1) all these features remain unchanged.

Now we turn to a completely different way of obtaining the overlap integral, a microscopic calculation based on Feynman's many-body wave function. We will show that there is a complete correspondence between the two approaches. This will enable us to determine the vortex mass, the Magnus force, and the coupling to the excitations. If $\psi_0(\mathbf{r}_1\cdots\mathbf{r}_N)$ is the ground state many-body wave function of He II, the system with a vortex may be described in a first approximation by [10]

$$|\psi(\mathbf{r}_0)\rangle=\prod_{j=1}^{N}\exp[i\theta(\mathbf{r}_j-\mathbf{r}_0)+\alpha(\mathbf{r}_j-\mathbf{r}_0)]\psi_0, \qquad (12)$$

where $\mathbf{r}_0$ is the vortex center, $\theta(\mathbf{r})$ the angle of $\mathbf{r}$, and $\alpha(\mathbf{r})$ a real function of $|\mathbf{r}|$. The most interesting feature of the wave function is that it changes phase by 2π whenever an atom moves around the vortex center once. In fact, it is by this feature that a vortex state should be defined; the above wave function should be regarded as an approximate description of the lowest energy state with this feature. The phase factors in (12) introduce a singularity at the vortex center, and this must be canceled by requiring $\exp[\alpha(\mathbf{r})]$ to vanish at the origin, otherwise the cost in kinetic energy would be too high. The particle density in the state, (12), therefore vanishes at $\mathbf{r}_0$. At large distances, the depletion of particle density due to the vortex vanishes like $|\mathbf{r}-\mathbf{r}_0|^{-2}$, and correspondingly α decays to zero like $|\mathbf{r}-\mathbf{r}_0|^{-1}$ [13].

The full calculation of the overlap integral from the many-body wave function is difficult, but we may expand $\ln O(\mathbf{r}_0,\mathbf{d})$ in powers of $\mathbf{d}$. In the small $|\mathbf{d}|$ limit the two leading terms will only involve one- and two-body density distributions in the state (12). Concrete results will then be obtained from a comparison with the same leading terms in (10). To facilitate the expansion we write

$$O(\mathbf{r}_0,\mathbf{d})=\langle\psi(\mathbf{r}_0')|\psi(\mathbf{r}_0)\rangle=\left\langle\exp\left\{\sum_j[ig_1(\mathbf{r}_j-\mathbf{r}_0,\mathbf{d})+g_2(\mathbf{r}_j-\mathbf{r}_0,\mathbf{d})]\right\}\right\rangle, \tag{13}$$

where $\langle\,\rangle$ denotes average in the state of (12), and $g_1(\mathbf{r},\mathbf{d})=[\theta(\mathbf{r}+\mathbf{d}/2)-\theta(\mathbf{r}-\mathbf{d}/2)]$ and $g_2(\mathbf{r},\mathbf{d})=\alpha(\mathbf{r}-\mathbf{d}/2)+\alpha(\mathbf{r}+\mathbf{d}/2)-2\alpha(\mathbf{r})$. Up to second order in $\mathbf{d}$, we may write $g_1(\mathbf{r},\mathbf{d})=\mathbf{d}\cdot\hat{\mathbf{z}}\times(\mathbf{r}-\mathbf{r}_0)/|\mathbf{r}-\mathbf{r}_0|^2$ and $g_2(\mathbf{r},\mathbf{d})=\frac{1}{4}(\mathbf{d}\cdot\nabla)^2\alpha(\mathbf{r})$. A straightforward cumulant expansion of (13) then yields, to the same order in $\mathbf{d}$, that

$$\ln O(\mathbf{r}_0,\mathbf{d})=\int d^2\mathbf{r}\rho(\mathbf{r})i\frac{\mathbf{d}\cdot\hat{\mathbf{z}}\times(\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|^2}+\int d^2\mathbf{r}\rho(\mathbf{r})\tfrac{1}{4}(\mathbf{d}\cdot\nabla)^2\alpha(\mathbf{r}-\mathbf{r}_0)-\frac{1}{2}\int d^2\mathbf{r}\rho(\mathbf{r})\left[\frac{\mathbf{d}\cdot\hat{\mathbf{z}}\times(\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|^2}\right]^2$$
$$-\frac{1}{2}\int\int d^2\mathbf{r}d^2\mathbf{r}'\rho(\mathbf{r},\mathbf{r}')\frac{\mathbf{d}\cdot\hat{\mathbf{z}}\times(\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|^2}\frac{\mathbf{d}\cdot\hat{\mathbf{z}}\times(\mathbf{r}'-\mathbf{r}_0)}{|\mathbf{r}'-\mathbf{r}_0|^2}+\frac{1}{2}\left[\int d^2\mathbf{r}\rho(\mathbf{r})\frac{\mathbf{d}\cdot\hat{\mathbf{z}}\times(\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|^2}\right]^2, \tag{14}$$

where $\rho(\mathbf{r})\equiv\langle\sum_j\delta(\mathbf{r}-\mathbf{r}_j)\rangle$ and $\rho(\mathbf{r},\mathbf{r}')\equiv\langle\sum_{i\neq j}\delta(\mathbf{r}-\mathbf{r}_i)\times\delta(\mathbf{r}'-\mathbf{r}_j)\rangle$ are the one- and two-body density distributions in the state (12).

The first order contribution to $\ln O(\mathbf{r}_0,\mathbf{d})$ in (14) is purely imaginary, which can be evaluated as $i\pi\rho_0\hat{\mathbf{z}}\cdot(\mathbf{d}\times\mathbf{r}_0)$ if we replace $\rho(\mathbf{r})$ by ρ_0, assuming that our system is confined within a disk centered at the origin of $\mathbf{r}$. The correction due to $\rho(\mathbf{r})-\rho_0\equiv\rho_1(\mathbf{r})$ is zero in the infinite size limit, because of the rotational symmetry in $\rho_1(\mathbf{r})$ about $\mathbf{r}_0$ and the fact that the density depletion decays sufficiently fast at large distances from $\mathbf{r}_0$. This first order term is the Berry phase associated with the Magnus force discussed in Ref. [14].

The second order contribution to $\ln O(\mathbf{r}_0,\mathbf{d})$ in (14) is purely real, and must also be negative as required by the fact that $|O(\mathbf{r}_0,\mathbf{d})|<1$ for nonzero $\mathbf{d}$. Therefore,

$$\ln O(\mathbf{r}_0,\mathbf{d})=i\pi\rho_0\hat{\mathbf{z}}\cdot(\mathbf{d}\times\mathbf{r}_0)-\frac{\mathbf{d}^2}{4l_d^2}+O(d^3), \tag{15}$$

where we have put the second order term as independent of the direction of $\mathbf{d}$ because of the isotropy of the system about the vortex center. The second order coefficient has been parametrized by a length l_d, which represents the same decay length as in (10).

We now examine closely the second order terms in (14), and show that their contribution to l_d^{-2} is finite. The term containing $\alpha(\mathbf{r}-\mathbf{r}_0)$ converges because the double derivative of α decays as $|\mathbf{r}-\mathbf{r}_0|^{-3}$ at large distances while $\rho(r)$ approaches a constant. At short distances, α may diverge like a logarithm, but $\rho(r)$ vanishes linearly, causing no trouble to the convergence of the integral. Therefore we shall no longer consider this term. In the presence of particle correlation, the form of $\rho(\mathbf{r},\mathbf{r}')$ is unknown for the state containing a vortex, except at large distances where it reduces to $\rho_0(\mathbf{r}-\mathbf{r}')$, the distribution in the absence of the vortex. We may replace the distributions by their asymptotic forms in (14) in order to examine the long distance contributions to these terms, because it is only from there that a divergence may ever be possible. Then, the last three terms of (14) [cf. (15)] yield

$$\frac{\mathbf{d}^2}{4l_d^2}=\frac{1}{2}\int\frac{d^2\mathbf{k}}{(2\pi)^2}\rho_0S_0(\mathbf{k})|F(\mathbf{k})|^2+\cdots, \tag{16}$$

where "$\cdots$" stands for the correction due to short distance contributions, $S_0(\mathbf{k})$ the static structure factor in the absence of the vortex, and $F(\mathbf{k})=i2\pi e^{i\mathbf{k}\cdot\mathbf{r}_0}\hat{\mathbf{z}}\cdot\mathbf{d}\times\mathbf{k}/k^2$ the Fourier transform of $\mathbf{d}\cdot\hat{\mathbf{z}}\times(\mathbf{r}-\mathbf{r}_0)/|\mathbf{r}-\mathbf{r}_0|^2$. It is known [13] that $S_0(\mathbf{k})=\hbar k/2Mc$ for small k, where M is the atomic mass and c the sound velocity. The integral in (16) therefore converges, meaning that the second order expansion in (14) exists in realistic situations.

Before we proceed further we comment on the validity of the above discussions. We have ignored multiparticle correlations induced by the vortex in the original wave function (12) and in the evaluation of the expression (14). We assume that the induced correlations decay sufficiently fast away from the vortex center, such that they do not affect the convergence properties at large distances. The situation at short distances is very complicated [10], and the short distance contribution can substantially reduce the overlap function. We expect, however, that the system should behave smoothly at short distances, so that no divergence can be induced from there. Our later arguments will only be based on the conclusion drawn above that the decay length l_d is finite.

The decay length l_d strongly depends on the interaction between the atoms in the superfluid. As the interatomic interaction becomes weak, the sound velocity decreases, which makes $S_0(\mathbf{k})$ large and therefore l_d small from (16). In the extreme case of no interatomic interaction, l_d becomes zero. This is just what one should expect from a direct calculation of (14) in the free boson limit, in which case $\rho(\mathbf{r},\mathbf{r}')=\rho(\mathbf{r})\rho(\mathbf{r}')(N-1)/N$.

With the overlap integral evaluated both from the effective theory, (10), and Feynman's many-body wave function, (15), we now would like to see how the parameters of the effective theory should be constrained. First, the parameter ρ_0 in the Magnus force of the effective theory is the same as the 2D superfluid density from the comparison of the results for the Berry phase term of the overlap integral. Second, in order to be consistent with the result of finite l_d from Feynman's many-body wave function, (16), the integral in (11) has to be convergent, implying that the spectral function $J(\omega)$ must vanish faster than ω at low frequencies and $|M(k)|$ must be less singular than k^{-1} at small k's. A comparison of (16) and (11) suggests that in the low frequency limit

$$J(\omega)=\frac{h\rho_0}{2Mc^2}\omega^2. \qquad (17)$$

In the language of quantum theory of dissipation [15], this kind of coupling is of the so-called super-Ohmic type. In a recent study of vortex tunneling in [16], a general heat bath is considered. It is found that a super-Ohmic coupling to the heat bath has a negligible effect on the tunneling process at low enough temperatures.

As for the mass of the vortex, our result of finite decay length implies that the mass of the vortex cannot be infinite; otherwise the localization length of the vortex would shrink to zero according to (8) and the decay length of the overlap function would become zero according to (11). Therefore, our result is consistent with that of Refs. [6,7], which suggest that m_v is zero or finite, and is in apparent disagreement with that of Ref. [8].

The vortex mass that we originally introduced in (1) may have already included the effect of renormalization by the polarization of all but the low lying excitations of the superfluid. There is still a possibility that it may be renormalized to infinity if the polarization of the low lying excitations is included. Indeed, if we neglect the Magnus force, a straightforward perturbative calculation [12] shows that the mass renormalization becomes logarithmically divergent if the coupling spectrum $J(\omega)$ goes as ω^2 at low frequencies. This is essentially the result in Ref. [8]. The divergence becomes more severe if $J(\omega)$ would vanish slower than ω^2.

The situation in the presence of the Magnus force is quite different. One can no longer set up a momentum eigenstate and extract an effective mass of the vortex from the momentum dependence of energy. A more natural approach is to relate the effective mass to higher Landau levels of cyclotron frequency $\omega_c=\hbar\rho_0/m_v$. Interaction with low lying excitations may shift and broaden the higher Landau levels, but these effects are not divergent in a perturbative calculation using (2). Therefore, if the higher Landau levels are well defined before turning on the coupling to the low lying excitations, we can conclude that further inclusion of such coupling has little effect on the higher Landau levels and thus the effective mass of the vortex. To observe a higher Landau level experimentally, one may trap ions in vortices produced in a rotating film of superfluid, and excite the vortices by electrical coupling to the ions [17].

Finally, we remark on the generality of our results. As long as the Feynman description of a vortex state is valid, everything else just follows from basic many-body physics such as the form of $S_0(\mathbf{k})$ at small k. As long as $S_0(\mathbf{k})$ vanishes with some positive power of k, orthogonality catastrophe in the overlap integral will not occur. Thus, our results may also be applicable to vortex structures in superconducting films and wire networks, Josephson junction arrays, and quantum spin systems.

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