Self-Consistent Theory of Mean-Field Electrodynamics

A. V. Gruzinov and P. H. Diamond
Department of Physics, University of California at San Diego, La Jolla, California 92093-0319
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Mean-field electrodynamics, including both α and β effects while accounting for the effects of small-scale magnetic fields, is derived for incompressible magnetohydrodynamics. The principal result is $\alpha=(\alpha_0+\beta_0\mathbf{R}\cdot\nabla\times\mathbf{R})/(1+R^2)$, $\beta=\beta_0$; where α_0,β_0 are conventional kinematic dynamo parameters, the reduction factor is proportional to the mean magnetic field $\mathbf{R}=R_m^{1/2}\mathbf{B}/(\rho V^2)^{1/2}$, R_m is the magnetic Reynolds number, and V is the characteristic turbulent velocity. This result follows from a generalization of the Zeldovich theorem to three dimensions, exploiting magnetic helicity balance.

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Helical motions of electrically conducting fluids generate magnetic fields. Examples include magnetic fields of the Earth, the Sun and stars, galactic [1,2] and accretion disk's fields [3], etc. The simplest theoretical paradigm for these fields is the kinematic dynamo theory [1]. The kinematic dynamo theory treats magnetic fields as passive—the magnetic field is distorted by the fluid motion, but the fluid itself does not feel the presence or tension of the magnetic field. In this approximation the problem of magnetic dynamo admits a convenient solution, what is called a kinematic dynamo theory [Eqs. (2) and (3) below]. The problem is, however, that being linear this theory fails to predict the strengths of generated fields. Below we derive a quasilinear version of the kinematic dynamo [quasilinear dynamo (QD)] which includes saturation effects associated with small-scale magnetic fields. In QD the small-scale magnetic fields (which are much stronger than the large-scale fields) modify the fluid motion in such a way as to reduce the α effect. This theory is capable of predicting magnitudes as well as spatiotemporal structure of the generated fields.

Conventional kinematic dynamo theory [1] studies diffusion and self-amplification of a passive vector field $\mathbf{B}$ in a given turbulent field $\mathbf{v}$:

$$\partial_t\mathbf{B}=\nabla\times(\mathbf{v}\times\mathbf{B})\,. \tag{1}$$

If we decompose magnetic field $\mathbf{B}$ into a large-scale field $\mathbf{B}_0$ and a small-scale field $\mathbf{b}$ (small-scale field is the part of the full magnetic field which is zero when averaged over the ensemble of turbulent fields), we get from (1)

$$\partial_t\mathbf{B}_0=\nabla\times\mathbf{E}\,, \tag{2}$$

$$\mathbf{E}\equiv\langle\mathbf{v}\times\mathbf{b}\rangle=\alpha\mathbf{B}_0-\beta\nabla\times\mathbf{B}_0\,, \tag{3}$$

where the dynamo (α effect) is proportional to the fluid helicity $\alpha=\alpha_0\equiv-\langle\mathbf{v}\cdot\nabla\times\mathbf{v}\rangle\tau$, and turbulent magnetic diffusivity (β effect) is proportional to the fluid energy $\beta=\beta_0\equiv\langle\mathbf{v}^2\rangle\tau$. Here, τ is the correlation time of the turbulence. Strictly speaking, we should write expressions for α_0 and β_0 as time integrals, but this symbolic notation is more clear and compact. The kinematic dynamo equations (2) and (3) predict an exponential growth of magnetic fields. It was always clear that when the magnetic field B_0 is amplified to the values comparable to the characteristic fluid turbulent velocity V (we set $4\pi\rho=1$) kinematic dynamo theory is invalid, and one should take into account backreactions of the magnetic field on the turbulent velocity field, thus leading to saturation of the generated magnetic fields. Recently Cattaneo and Vainshtein [4] pointed out that the above naive estimate for the applicability of conventional theory $B_0<V$ is incorrect. The physical reason for that is the following. The Zeldovich estimate [5] for the small-scale field energy (in two dimensions) is $\langle b^2\rangle\sim R_mB_0^2$, where $R_m\gg1$ is the magnetic Reynolds number. This estimate is valid in two dimensions and follows from conservation of the squared magnetic potential in dissipationless magnetohydrodynamics (MHD). Thus before the large-scale field energy becomes comparable to the kinetic energy of turbulence, we reach the state with the small-scale field in energetic equipartition with turbulence. We may suppose that conventional kinematic dynamo theory is applicable when $\langle b^2\rangle<V^2$, that is, only for extremely small values of the large-scale fields $B_0<R_m^{-1/2}V$. Note, however, that the Zeldovich theorem is applicable only in two dimensions, thus explaining why Cattaneo and Vainshtein were able to calculate magnetic diffusivity suppression only for two dimensions. We show below that the estimate $B_0<R_m^{-1/2}V$ for applicability of the kinematic dynamo theory is also valid in the three dimensional MHD. However, the dynamics when B_0 reaches values close to or greater than $R_m^{-1/2}V$ are more complicated in three dimensions than in two.

The kinematic dynamo equation (3), being just a Taylor expansion in k space, remains valid. However, the dynamo process is modified in a rather complicated fashion,

$$\alpha=(\alpha_0+\beta_0\mathbf{R}\cdot\nabla\times\mathbf{R})/(1+R^2)\,, \tag{4}$$

where the reduction factor $\mathbf{R}$ is just the large-scale field B measured in the units of $(\rho V^2/R_m)^{1/2}$. Magnetic diffusivity survives in its kinematic form $\beta=\beta_0$. Equations (3) and (4) constitute what we call quasilinear dynamo (QD). The structure of (2)–(4) is somewhat reminiscent

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of the result of [6], where Kraichnan considered the effects of a fluctuating α. In QD generated magnetic fields saturate due to the α-effect suppression. Below we briefly describe some critical points in the derivation of the QD equations.

We first note that Eqs. (2) and (3) remain valid for arbitrary values of the large-scale field B_0. However, now we are able to calculate the average $\mathbf{E}$ using both (1) and the fluid equation describing backreactions:

$$\partial_t \mathbf{v} = (\mathbf{B}_0 \nabla)\mathbf{b} + (\mathbf{b}\nabla)\mathbf{B}_0 - \nabla p\,, \tag{5}$$

where $\nabla\cdot\mathbf{v}=0$. Now we should calculate the average electric field $\mathbf{E}$ using

$$\mathbf{E} = \left\langle \mathbf{v}\times\int dt\,\partial_t \mathbf{b}\right\rangle - \left\langle \mathbf{b}\times\int dt\,\partial_t \mathbf{v}\right\rangle. \tag{6}$$

The last term in (6) is quadratic in $\mathbf{b}$ and contains a part which is linear in $\mathbf{B}_0$ (giving a quasilinear correction to α) and a part which is linear in $\nabla\times\mathbf{B}_0$ (giving a correction to β). Thus, it turns out that one can obtain the correct result for the α effect rewriting (5) as if $\mathbf{B}_0$ were a constant vector, i.e., $\partial_t\mathbf{v} = (\mathbf{B}_0\nabla)\mathbf{b}$. We then get

$$\alpha = -\tau(\langle \mathbf{v}\cdot\nabla\times\mathbf{v}\rangle - \langle \mathbf{b}\cdot\nabla\times\mathbf{b}\rangle)\,. \tag{7}$$

This last formula was first obtained in [7]. Note that it sensibly predicts that the α effect is reduced by the average $\langle \mathbf{b}\cdot\nabla\times\mathbf{b}\rangle$ (which is related to but is not magnetic helicity), and α is zero in a "pure Elsasser" or Alfvén wave state $\mathbf{v}=\pm\mathbf{b}$.

Now we should calculate the average $\langle \mathbf{b}\cdot\nabla\times\mathbf{b}\rangle$. This can be done exactly. The reason why is the existence of the magnetic helicity integral of motion for ideal MHD. The following procedure, based on helicity balance, is in fact a three dimensional analog of the original Zeldovich theorem [5]. We just write the equation for the small-scale field in a certain gauge in terms of vector potential $\mathbf{a}$,

$$\partial_t \mathbf{a} = \mathbf{v}\times\mathbf{B}_0 + \mathbf{v}\times\mathbf{b} - \langle \mathbf{v}\times\mathbf{b}\rangle - \eta\nabla\times\mathbf{b}\,, \tag{8}$$

where η is the "molecular" magnetic diffusivity due to finite conductivity. Multiplying Eq. (8) by $\mathbf{b}$ and taking the average gives $\langle \mathbf{b}\cdot\mathbf{v}\times\mathbf{B}_0\rangle = \eta\langle \mathbf{b}\cdot\nabla\times\mathbf{b}\rangle$. The term on the left hand side (lhs) of the last equation is calculated using the definition of the average field $\mathbf{E}$:

$$\langle \mathbf{b}\cdot\mathbf{v}\times\mathbf{B}_0\rangle = -\mathbf{B}_0\cdot\langle \mathbf{v}\times\mathbf{b}\rangle = -\mathbf{B}_0\cdot\mathbf{E}\,,$$

and then (3) gives

$$\langle \mathbf{b}\cdot\nabla\times\mathbf{b}\rangle = -(\alpha/\eta)B_0^2 - (\beta/\eta)\mathbf{B}_0\cdot\nabla\times\mathbf{B}_0\,. \tag{9}$$

We finally plug (9) into (7) to get the QD equation (4). Equation (9) is the topologically motivated, exact expression for $\langle \mathbf{b}\cdot\nabla\times\mathbf{b}\rangle$. Since this is a central point of the QD theory, we wish to give an alternative derivation for the average (9). Namely, we simply note that magnetic helicity dissipation is due to molecular diffusivity only, so that

$$\partial_t\langle \mathbf{A}\cdot\mathbf{B}\rangle = -2\eta\langle \mathbf{B}\cdot\nabla\times\mathbf{B}\rangle\,. \tag{10}$$

The lhs of (10) can be calculated with $\mathbf{A},\mathbf{B}$ replaced by the large-scale fields $\mathbf{A}_0,\mathbf{B}_0$, while the rhs can be calculated with $\mathbf{B}$ replaced by the small-scale field $\mathbf{b}$. Equation (10) then becomes

$$\partial_t\langle \mathbf{A}_0\cdot\mathbf{B}_0\rangle = -2\eta\langle \mathbf{b}\cdot\nabla\times\mathbf{b}\rangle\,. \tag{11}$$

Now we use (3) to calculate the lhs of (11) and we again obtain (9).

It remains to prove that the β effect is not renormalized in three dimensions. It turns out to be a more complicated task than to calculate the renormalized α. The technical problem is the presence of the term ∇p in Eq. (5). It was possible to omit this term in the course of α calculation, but it should be kept for the β calculation. We are thus forced to use the spatial Fourier transformation. Note also that β is unrenormalized only in exactly three dimensions. To point out this d dependence and to show that $\beta=\beta_0$ is just a consequence of three dimensionality we calculate the quasilinear β in d dimensions. The basic line of this calculation is straightforward—write Eqs. (5) and (6) in spatial Fourier components and do the average (6). Since the calculation involves too much algebra, we simply give the result and leave the details for the extended version of the present paper. The renormalized turbulent magnetic diffusivity in d dimensions is given by the following analog of (7):

$$\beta = \tau[\langle v^2\rangle + (2-6/d)\langle b^2\rangle]\,. \tag{12}$$

Thus we see that for $d=2$ we get β-effect suppression [2], while for $d=3$ the β effect is not modified by backreactions. We do not need to know the small-scale field energy $\langle b^2\rangle$ in three dimensions. The cross helicity correlator $\langle \mathbf{v}\cdot\mathbf{b}\rangle$ is also irrelevant for the QD derivation.

To summarize, the large-scale magnetic field in a helical small-scale turbulent field of electrically conducting fluid is described by the quasilinear dynamo equations

$$\partial_t\mathbf{B} = \nabla\times(\alpha\mathbf{B} - \beta_0\nabla\times\mathbf{B})\,, \tag{13}$$

$$\alpha = (\alpha_0 + \beta_0\mathbf{B}\cdot\nabla\times\mathbf{B})/(1+B^2)\,, \tag{14}$$

where B is in Zeldovich units $(\rho V^2/R_m)^{1/2}$. The model can predict not only the frequencies and length scales of the generated fields but also their saturation levels. The actual saturation level of the generated field B depends on the geometry of the problem, and is equal to a geometrical factor times the "Zeldovich" unit magnetic field $(\rho V^2/R_m)^{1/2}$.

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[1] H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge Univ. Press, Cambridge, 1978).
[2] E. N. Parker, *Cosmical Magnetic Fields* (Clarendon Press, Oxford, 1979).
[3] E. T. Vishniac, L. Jin, and P. H. Diamond, Astrophys. J. **365**, 648 (1990).
[4] F. Cattaneo and S. I. Vainshtein, Astrophys. J. **376**, L21 (1991).
[5] Ya. B. Zeldovich, Sov. Phys. JETP **4**, 460 (1957).
[6] R. H. Kraichnan, J. Fluid Mech. **75**, 657 (1976).
[7] A. Pouqet, U. Frisch, and J. Leorat, J. Fluid Mech. **77**, 321 (1976).