Stabilization of Chaos by Proportional Pulses in the System Variables

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We present a new method that allows one to stabilize chaotic systems by applying proportional pulses to the system variables. The method does not require any previous knowledge of the system dynamics. Acting on the system variables may be most useful for chemical and biological systems, situations for which in some cases it may be difficult to find an accessible system parameter.

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In recent years many low-dimensional nonlinear dynamical systems have been seen to show the kind of behavior known as deterministic chaos, being characterized as chaotic systems by their strong sensitivity to small perturbations in their initial conditions. Instead, in many practical situations one is interested in obtaining a more regular behavior for the system, although the system might need to be changed considerably. Yet, Ott, Grebogi, and Yorke (OGY) have shown in a seminal contribution [1] that it is possible to obtain a regular, or periodic, behavior by making only *small* time-dependent perturbations in some accessible *system parameters*. A number of experimental studies have shown how to implement these ideas in practice in different fields [2–6] (see also [7] for a survey).

The OGY method [1] and its variants [8–10] work by slightly perturbing a system parameter such that the system remains attracted by the stable manifold of one of the underlying unstable periodic orbits coexisting with the strange attractor. In practice, one first tracks the desired orbit and then applies the necessary changes in the system parameter depending on the deviation of the system from this orbit. There exist other (nonfeedback) methods in which one does not perform changes in the system according to its position in phase space, applying, instead, weak periodic perturbations on some system parameters [11–14].

In this Letter we shall consider the possibility of stabilizing chaotic systems by performing rather nonspecific changes in the system variables, and not in system parameters. This possibility might be of application in those situations where finding an accessible system parameter may be a difficult task, namely, the case of certain chemical, biological, or spatially nonhomogeneous systems.

In the present method one performs changes in the system variables in the form of instantaneous pulses spaced in time, namely, every interval τ, in the form

$$X_i'(t) = X_i(t)[1 + \lambda], \tag{1}$$

where $X_i(t)$ represents the ith variable of the system and λ, which can be positive or negative, regulates the strength of the pulse. One can think of a practical implementation of this idea in the case of a batch chemical reactor consisting of the injection of some amount of either an inert compound or one of the i components.

As an application, we have stabilized a dynamical system with three variables: the isothermal autocatalator model of Peng, Scott, and Showalter [15], a model that, in spite of its simplicity, is able to show a number of features observed in the chemical experiments, including deterministic chaos. All the numerical work has been performed by employing a fourth-order Runge-Kutta algorithm [16], where the step length has been carefully chosen in order to avoid spurious behavior. From the numerical work some features of the method have arisen. First of all, there is a relationship between the two parameters of (1), the intensity of the pulses λ and the time elapsed between these pulses τ, such that these two quantities are related in the form $\lambda/\tau = \text{const}$; this relationship is only approximate for high values of τ. This property implies that once the behavior of the method is known for a given value of τ it is possible to extrapolate to any other value.

On the other hand, the method shows the usual routes to chaos as a function of λ. In other words, by varying λ one can switch from a period-1 (limit cycle) behavior to a chaotic behavior (in the limit of no perturbation, i.e., of low values of λ) following the usual subharmonic cascade, and Feigenbaum's constant $\delta = 4.66\ldots$ is obtained in the limit of period doubling bifurcations. This implies that the present method works by creating a *new* dynamical system that has λ as a system parameter. In Figs. 1 and 2 we present some results for this model with λ negative, where different limit cycles have been stabilized, namely, one- and two-cycles, although four- and eight-cycles can also be found easily. A vertical dashed line splits the time series in two parts: In the first one the system is driven by the deterministic equations alone, while in the second one the stabilization algorithm (1) is operating. The parameter values considered are close to the corresponding bifurcation values, for in the sense that if a slightly lower value of λ were used, then one would obtain a cycle with

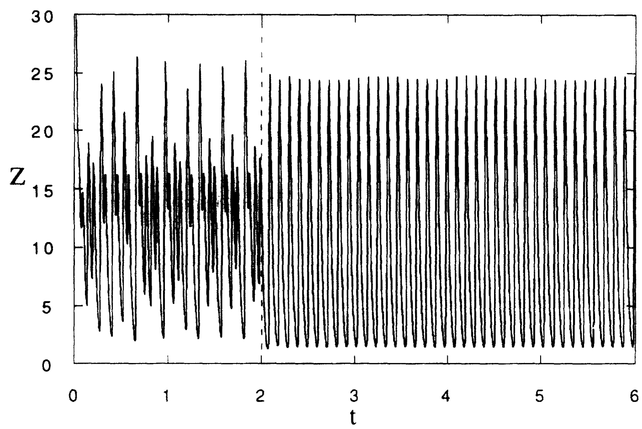

FIG. 1. A stabilized period-1 orbit of the three-variable autocatalator model [15]; the values for the parameters considered here are $\kappa = 10$, $\delta = 0.02$, $\sigma = 0.005$, and $\mu = 0.154$. The stabilization acts after $t = 2$. The integration step has been taken as $\Delta t = 0.00005$ and the parameters for (1) are $\lambda = -0.02$ and $\tau = 100\Delta t$.

doubled periodicity.

It is possible to obtain a periodic behavior by using both positive and negative values of λ, although the resulting limit cycles may be different. This happens, in particular, for the results shown in Fig. 3. For high values of $|\lambda|$ the system is stabilized in the form of a fixed point, although the behavior of the system is different for the two cases of positive and negative values of λ. For the $\lambda > 0$ case the stabilized fixed point is close, although not equal, to an unstable fixed point of the original system [17], and the corresponding one-cycle originates from the latter through a Hopf-like bifurcation. Then, the limit cycle shrinks in phase space as one goes to high values of λ. Instead, in the $\lambda < 0$ case the system appears to perform a transition from the limit cycle to a newly formed fixed point [18].

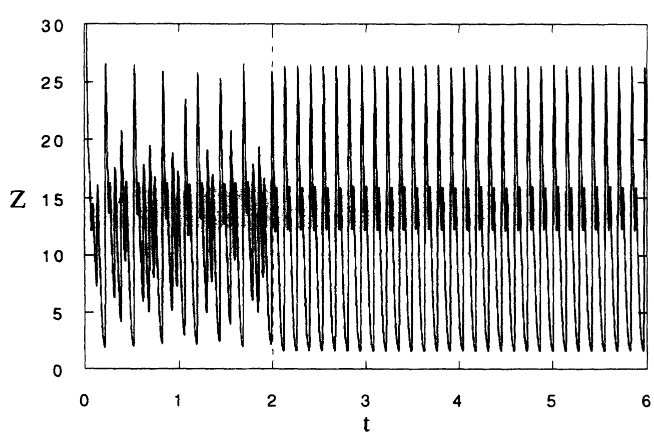

FIG. 2. A stabilized period-2 orbit of the three-variable autocatalator model. The parameters for (1) are $\lambda = -0.006$ and $\tau = 100\Delta t$. For further details see Fig. 1.

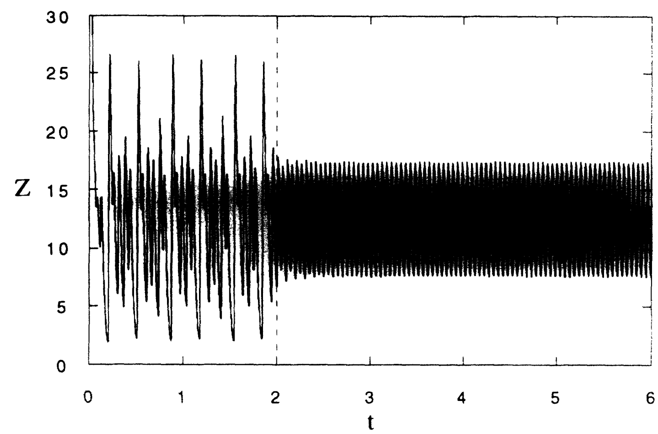

FIG. 3. A stabilized period-1 orbit of the three-variable autocatalator model. The parameters for (1) are $\lambda = 0.006$ and $\tau = 100\Delta t$. For further details see Fig. 1.

We have been able to stabilize a number of continuous dynamical systems, like the Rössler and Lorenz models. In this context it is interesting to deal with the Lorenz model [19], because for $\lambda < 0$ the present method stabilizes in this case one of the two unstable fixed points around which the system spirals in phase space [20]. Figure 4 shows a time series being stabilized by the system in the fixed point with x and y negative. Depending on the initial conditions it is possible to stabilize the other fixed point with x and y both positive. In the case of $\lambda > 0$ the system is described by a limit cycle around the third unstable fixed point (at the origin) of the original system. For high enough values of λ the system tends to go to infinity. This unbound behavior for $\lambda > 0$ has also been found to occur for the Rössler model.

The phase space trajectory actually followed if a limit cycle is stabilized consists of a series of continuous lines

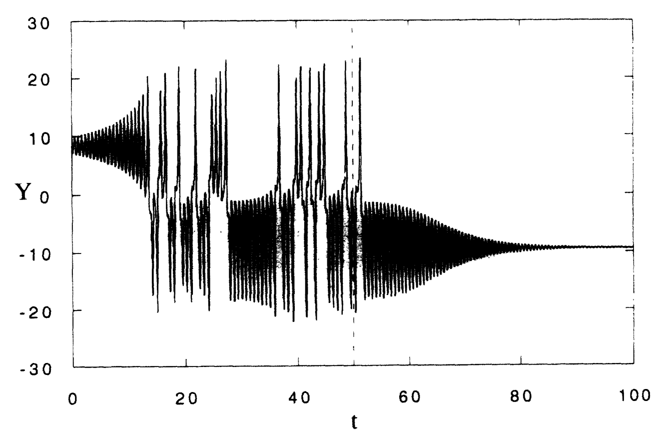

FIG. 4. A stabilized fixed point for the Lorenz model [19]. The values for the parameters considered here are $\sigma = 10$, $\beta = 8/3$, and $\rho = 28$. The stabilization acts after $t = 50$. The integration step has been taken as $\Delta t = 100\Delta t$ and the parameters for (1) are $\lambda = -0.02$ and $\tau = 0.005$.

that present discontinuities at the points where the pulses are acting. One can think of a smooth stable orbit that in some sense *shadows* the stabilized trajectory. An important question is whether this *shadow* orbit, belonging to the stabilized new dynamical system, corresponds to some unstable periodic orbit of the original system. In the already discussed situations where the system evolution is described by a fixed point, it is found that for some cases, i.e., Lorenz with $\lambda < 0$ and autocatalator with $\lambda > 0$, the stabilized fixed point appears to be close to an unstable fixed point of the original system. Instead, the Lorenz system with $\lambda > 0$ has a tendency to go to the fixed point at the origin, although the evolution is unbound, while in the $\lambda < 0$ case of the autocatalator model the algorithm creates a new fixed point at the origin. In conclusion, the stabilization algorithm appears to stabilize in some cases, although not in all of them, fixed points close to those of the original system.

One can try to relate our results with those obtained by some authors, observing that some particular numerical methods, e.g., backward Euler, can suppress chaos [21]. In fact, computers introduce through rounding errors changes in the variables that are not constant, but rather depend on the value of the variables. Perhaps in this *numerical* context it might be possible to understand the relationship between λ and τ if the error in each time step is additive.

An important issue in a method that pretends to stabilize chaotic systems is its robustness versus the application of external noise. We shall consider first the application of multiplicative white noise in the form

$$X_i' = X_i + \sigma[0,1]\,\gamma X_i \tag{2}$$

at every integration step Δt, where $\sigma[0,1]$ is a stochastic variable of zero mean and a Gaussian distribution with width 1. The maximum allowable intensity of noise γ is related to the intensity of the pulses λ for $\tau = \Delta t$, that establishes a well defined scale. In Fig. 5 we show an example of the behavior of the system when this noise (2) is applied. Things are not so clear when one has additive noise (of the type $X_i' = X_i + \sigma[0,1]\,\gamma$). In this case the effect of noise can be very important for those variables that have a very low value in some cycles, and less can be said in principle. In Fig. 6 we show an example of one of these stochastic trajectories. It can be seen that tiny perturbations when the value of the variable is low induce sometimes transient behaviors in the following cycle.

In conclusion, in this Letter we have introduced a new method that allows the stabilization of a chaotic system by acting on the system variables, rather than in some available parameter. The method is implemented by applying pulses that change proportionally the system variables. In a practical application, for instance in the case of chemical or biological systems, this method could be implemented by performing a dilution in the system, i.e., by making the concentration of the species diminish by some amount.

Among the practical consequences of stabilizing chaotic systems in this way one may mention the possibility of characterizing better chaotic states. It is often argued that deterministic chaotic behavior in a time series cannot be easily distinguished from the presence of purely stochastic behavior. The sensitivity of a system supposed to be chaotic to this simple algorithm that performs changes in the system variables could afford a way of achieving this distinction.

Control and stabilization of chaotic systems allow one to look at nonlinear systems with a different perspective, as the chaotic behavior can offer more opportunities for biological systems to react to a changing environment. The reason is that a chaotic system has a virtually unlimited reservoir of periodic behavior. Living beings

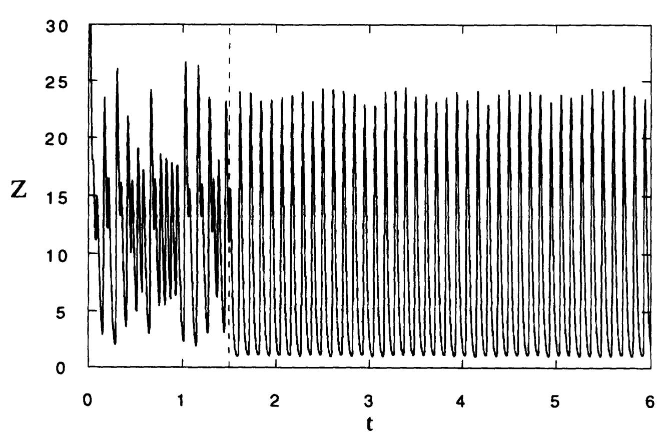

FIG. 5. The effect of multiplicative noise (2) is shown for the three variable autocatalator model. The intensity of the applied noise is $\gamma = 0.001$, and the parameters of the stabilization algorithm are $\lambda = -0.003$ and $\tau = 10\Delta t$. For further details see Fig. 1.

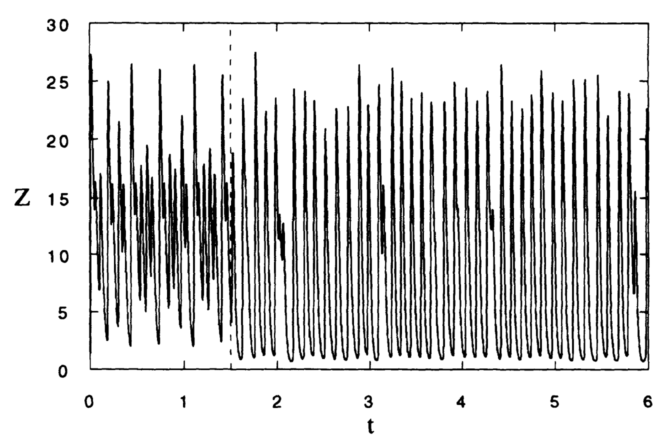

FIG. 6. The effect of additive noise is shown for the three variable autocatalator model. The intensity of the applied noise is $\gamma = 0.002$, and the parameters of the stabilization algorithm are $\lambda = -0.003$ and $\tau = 10\Delta t$. For further details see Fig. 1.

are self-controlling biological systems, and pulses in the neural system provoking pulses of some chemical control substances may be an attractive mechanism for switching among different periodic states. Stabilized chaotic systems could be at the edge between order and chaos, something that some biologists claim to be necessary for complexity to occur [22].

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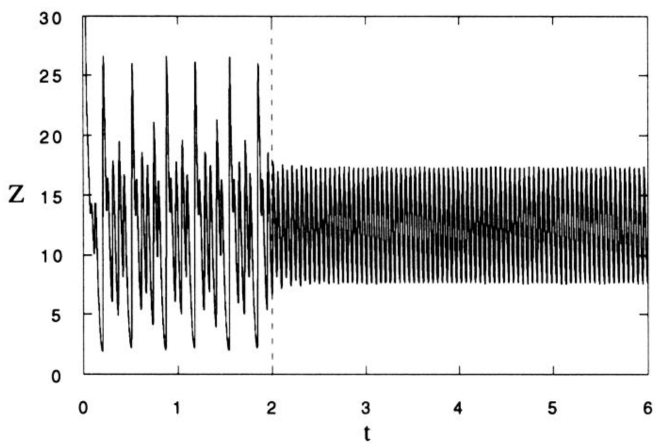

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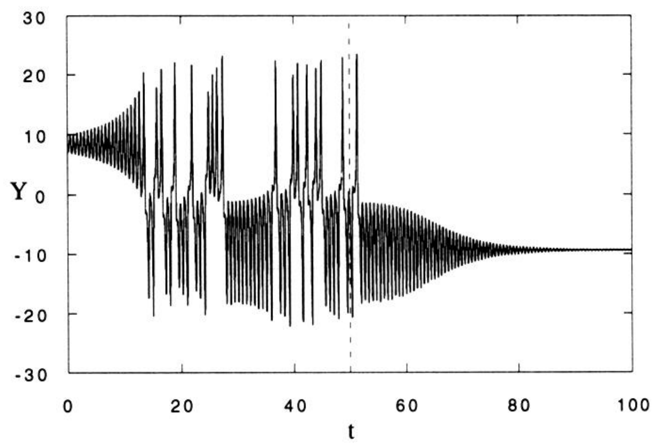

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