Chaos, Noise, and Synchronization

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We show that a pair of chaotic systems subjected to the same noise may undergo a transition at large enough noise amplitude and follow almost identical trajectories with complete insensitivity to initial conditions. An analytic argument is presented to show that a pair of generic systems in the same potential evolving to equilibrium through standard Langevin dynamics with the same noise collapse into the same trajectory at long times.

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A generic feature of the chaotic state of nonlinear dynamical problems is the extreme sensitivity to initial conditions of the trajectory of a particle [1]. In this Letter we show, in several different contexts, that two chaotic systems linked with a common noise term, with sufficient strength, have trajectories that coalesce and become identical (albeit remaining noisy) at long times. Since, in a chaotic state, deterministic systems exhibit an apparent random behavior, one might have expected that turning on actual randomness might lead to the system becoming even more "random." Our results are counter to this intuitive expectation. This new phenomenon is distinct from the synchronization in chaotic systems wherein certain subsystems of nonlinear chaotic systems whose Lyapunov exponents are negative will synchronize when they are linked by common signals [2].

The phenomenon is best described in the context of a one dimensional logistic map [1]. Specifically, we consider the trajectories of a noisy logistic map where the position at (discrete) time $t+1,x'$, depends on the position at time t,x through the equation

$$x'=4x(1-x)+\eta\,, \tag{1}$$

where $0<x<1$ and η is a random number chosen uniformly from the interval $-W$ to $+W$ with the constraint that $0<x'<1$. Thus if a chosen η violates the bounds $0<x'<1$, it is discarded and a new η is chosen. We find the remarkable result that for $W>W_c$ (we estimate $W_c\simeq\frac{1}{2}$) the trajectory for various initial conditions becomes point by point identical in time (up to the precision of the computer) ultimately following a single random trajectory, when an identical sequence of η's is used independent of the initial condition. Figure 1 shows the mean square distance between pairs of identically driven (i.e., having the same noise) randomly chosen initial conditions as a function of the number of iterations of the logistic map. Three values of W were chosen: $W=0$ for the first 10^6 iterations, $W=0.3$ for the next 10^6, and $W=0.6$ for the final 10^6 iterations. The results vividly illustrate that when $W>W_c$, the trajectories of the particles become identical at long enough times; i.e., they are synchronized.

In order to characterize the transition occurring at W_c, we have also studied the probability density distribution $P(d)$ at long times (5×10^5 iterations with double precision) averaged over a hundred randomly chosen initial pairs, where $d=|x-y|$ and x and y are the pair of numbers being iterated. Specifically, we choose a pair of random numbers, each of which is iterated 5×10^5 times with the *same* noise. The values of d are determined at this time. The calculation is then repeated 100 times to get a probability distribution for d.

Our results indicate that, for a finite precision ε (when d becomes smaller than ε, it is set to zero and remains zero thereafter),

$$P(d)=(1-\lambda)\bar{P}(d)+\lambda\delta(d)\,, \tag{2}$$

where δ is the Dirac delta function, the normalized $\bar{P}(d)$

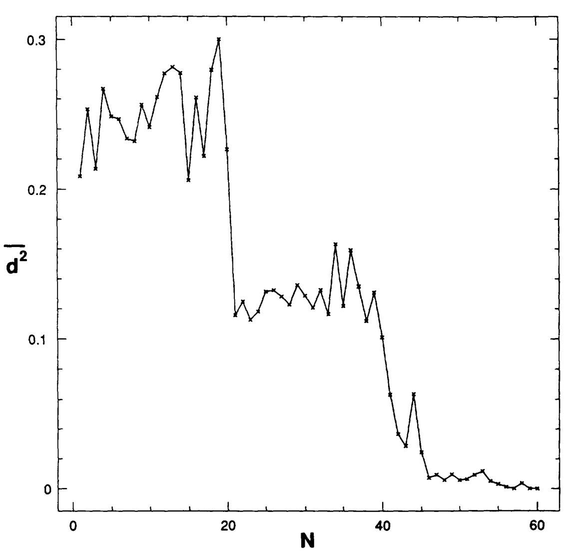

FIG. 1. Plot of mean square distance versus number of iterations (each unit represents 50000 iterations) for the logistic map. $W=0$ for the first 10^6 iterations; $W=0.3$ for the next 5×10^5 iterations; and $W=0.6$ for the last 5×10^5 iterations. The calculations were carried out in double precision. The results were averaged over 100 realizations. In single precision, the mean square distance goes to zero more rapidly.

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is a smooth function of W, and $\lambda(W)$ is shown in Fig. 2. In the limit of infinite time and finite precision, λ would be equal to 1 for $W > W_c$. (More precisely, one may work with perfect precision and consider the fraction of systems with $d < \varepsilon$ at time t. λ is then the limiting value of this fraction on first letting $t \to \infty$ and then letting $\varepsilon \to 0$.)

Thus, in terms of λ we have a transition at $W = W_c$. $\lambda(W)$ is identically zero for $W < W_c$ and it is different from zero for $W > W_c$. Note that for finite time and finite precision λ does not increase monotonically with W—it has a maximum value around $W \approx 0.55$. Our results for the logistic map may be understood qualitatively as arising from the extreme deamplification of the distance between neighboring points when the sum of the pair of numbers is 1 since the distance evolves according to

$$d' = 4d\{1 - (x+y)\} . \tag{3}$$

Notice that stochasticity due to the noise enters Eq. (3) through $x+y$ whose evolution does depend on η. Since the density of successive iterates of the logistic map is high close to 0 and 1, a W greater than or equal to $\frac{1}{2}$ allows both the numbers to be shifted in such a way that they are both close to $\frac{1}{2}$ and their sum becomes close to 1.

We have verified that this phenomenon is robust to changes in the coefficient 4 in the logistic map [Eq. (1)]. Note also that because the logistic map exhibits a high density of states near 0 or 1, the added noise (with the constraint that the pair of numbers stay within the basin of attraction) is asymmetric.

In order to underscore the generality of the phenomenon, we now turn to the Lorenz equations [1]

$$\frac{dx}{dt} = Py - Px, \quad \frac{dy}{dt} = -xz + rx - y, \quad \frac{dz}{dt} = xy - bz , \tag{4}$$

with $P = 10$, $b = \frac{8}{3}$, and $r = 28$, which we integrate with a time step of 0.001 unit. Unlike the logistic map, the Lorenz equations have an unbounded basin of attraction for the strange attractor. We now generalize the y equation [3] (in difference form) to

$$y(t+\Delta t) = y(t) + [-x(t)z(t) + rx(t) - y(t)]\Delta t + (\text{random}) W_L \sqrt{\Delta t} , \tag{5}$$

where (random) denotes a random number generated from a uniform distribution between 0 and 1 and the amplitude W_L plays the role of W in the logistic map. Unlike in the logistic map, which has a finite basin of attraction, the noise term is not peculiar here. (In the logistic map, the noise had to be chosen specially in order to ensure that the iterate stayed within the basin of attraction. Thus the noise term depended on the state of the system.) The $\sqrt{\Delta t}$ in (5) has been introduced in order that the threshold scales independently of Δt in the $\Delta t \to 0$ limit. A threshold value of the amplitude ($\sim \frac{2}{3}$) is found beyond which one observes the synchronization effect as in the logistic map (Fig. 3). This quenching does not take place (for $W_L = 1$ within 10^6 time steps) when the noise term with the same amplitude is made symmetric [i.e., random is replaced by (random $-$ 0.5)].

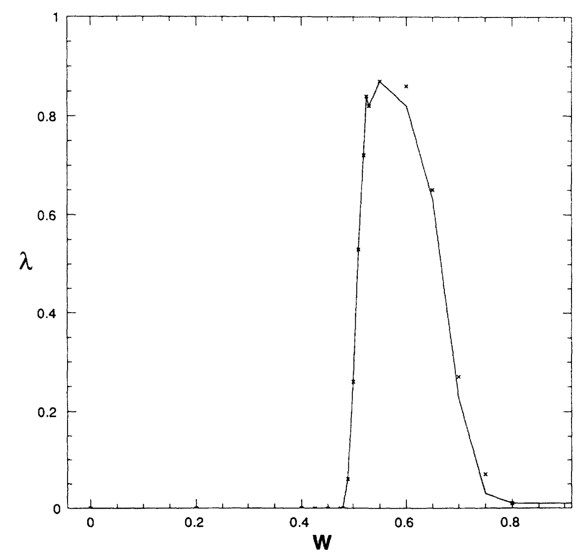

FIG. 2. Plot of λ, defined in Eq. (2), versus W. The results were obtained after 500000 iterations in double precision and used 100 independent runs. The solid line consists of straight line segments through points obtained using the criterion that $(x-y)^2$ is equal to zero in double precision. The crosses denote a criterion of $|x-y| < 10^{-20}$.

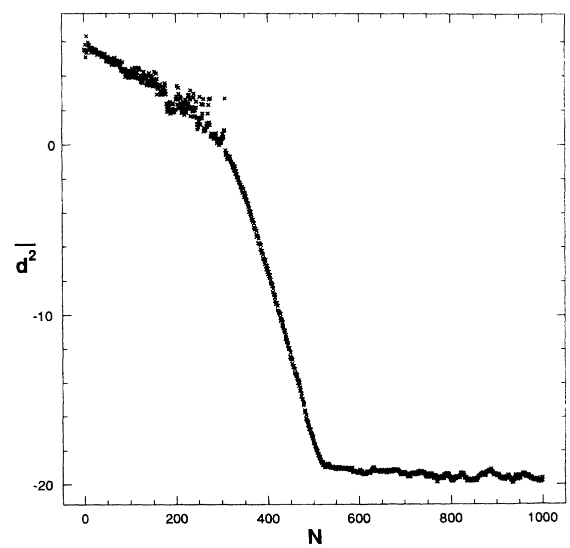

FIG. 3. Plot of mean square distance versus number of iterations (each unit represents 1000 iterations) for the Lorenz equations. The noise amplitude $W_L = 0.7$. The calculations were carried out in single precision and the results averaged over 100 realizations.

Several authors have considered the role played by noise in bringing about a transition from chaotic behavior to "ordered" behavior. Matsumoto and Tsuda [4] report studies of "noise induced order" caused by smearing and shifts of the peaks in the invariant density—a peak near the fixed point in the absence of noise moves to a region close to the critical point due to the noise. Such a transition was observed in a model of the Belousov-Zhabotinsky reaction but *not* in the logistic map [4]. Recently, a great deal of progress has been achieved in the "control" of chaos [5,6]. The key idea is that the chaotic motion of a dynamical system can be converted into a periodic motion by imposing tiny carefully chosen time dependent perturbations of a system parameter to enforce one of the unstable orbits embedded within the strange attractor. An alternative method of Hubler and coworkers [7,8] does not use a feedback mechanism but relies on the construction of model equations for the dynamics and modifies the driving force to ensure that the desired dynamics is the stable solution. A possible connection between the phenomena discussed here and the results of Refs. [4,7,8] is the presence of large regions with small local Lyapunov exponents that cause synchronization to goal orbits in convergent areas. A simple way of viewing our results is to visualize the noisy chaotic equations as arising from the coupling of the chaotic system (with a positive Lyapunov exponent) with other variables (constituting the noise) so that the supersystem has a negative Lyapunov exponent [9]. However, the strange attractor is *not* replaced by anything simple such as a fixed point or limit cycle.

Fahy and Hamann (FH) [10,11] demonstrated, based on numerical studies, that when particles with different initial conditions were driven by an identical sequence of random forces, their trajectories may become identical at long times if the time interval between successive random forces becomes smaller than a given threshold. This result has important consequences in Monte Carlo applications [10]. We now turn to a simple analytic argument for the FH result. We will see that this phenomenon is of the same type exemplified in the two systems described above.

In the limit in which the time separation between successive forces becomes small compared to any characteristic macroscopic time scale of the problem the FH effect can be understood in terms of Langevin dynamics [12]. Indeed let us consider a pair of particles with different initial locations moving in the *same* d-dimensional potential subject to the *same* noise governed by Langevin equations of the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + \boldsymbol{\eta}(t) \tag{6}$$

and

$$\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y}) + \boldsymbol{\eta}(t), \tag{7}$$

with $\langle \eta_\alpha(t)\eta_\beta(t')\rangle = 2k_BT\delta_{\alpha\beta}\delta(t-t')$, $\alpha = 1,2,\ldots,d$. If $\mathbf{F}(\mathbf{x}) = -\nabla_x V(\mathbf{x})$, the probability distributions of the positions of the particles at infinite time will be $P_{\text{eq}}(\mathbf{x}) \propto e^{-\beta V(\mathbf{x})}$ provided the system is ergodic. We now turn to the Fokker-Plack (FP) [12] equation satisfied by the joint probability distribution $P(\mathbf{x},\mathbf{y},t)$,

$$\dot{P} = -\sum_\alpha \left[\frac{\partial}{\partial x_\alpha}[F_\alpha(\mathbf{x})P] + \frac{\partial}{\partial y_\alpha}[F_\alpha(\mathbf{y})P]\right] + k_BT(\nabla_x^2 + \nabla_y^2 + 2\nabla_x\cdot\nabla_y)P,$$

where $\nabla_x = [\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_d]$ and the initial condition is $P(\mathbf{x},\mathbf{y},t=0) = \delta(\mathbf{x}-\mathbf{x}_0)\delta(\mathbf{y}-\mathbf{y}_0)$. The above equation can be conveniently recast in the form $\dot{P}(\mathbf{r},t) = -\sum_{i=1}^{2d}\partial_i J_i$, with $J_i = f_i P - k_BT\sum_{j=1}^{2d}\Gamma_{ij}\partial_j P$, where $\mathbf{f}$, $\mathbf{r}$, and $\boldsymbol{\partial}$ are 2D component vectors,

$$\mathbf{f} = (F_1(\mathbf{x}), F_2(\mathbf{x}), \ldots, F_d(\mathbf{x}), F_1(\mathbf{y}), F_2(\mathbf{y}), \ldots, F_d(\mathbf{y})),$$

$$\mathbf{r} = (x_1, x_2, \ldots, x_d, y_1, y_2, \ldots, y_d),$$

and

$$\boldsymbol{\partial} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \cdots \frac{\partial}{\partial x_d}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} \cdots \frac{\partial}{\partial y_d}\right);$$

$\Gamma_{ij} = 1$ when $i = j$ or $|i-j| = d$ and zero otherwise. The stationary solution of the FP equation [the $t \to \infty$ limit of $P(\mathbf{x},\mathbf{y},t)$] will be such that $J_i = 0$ for $i = 1,\ldots,2d$, implying that

$$\mathbf{F}(\mathbf{x})P_s(\mathbf{x},\mathbf{y}) - k_BT(\nabla_x + \nabla_y)P_s(\mathbf{x},\mathbf{y}) = 0 \tag{8}$$

and

$$\mathbf{F}(\mathbf{y})P_s(\mathbf{x},\mathbf{y}) - k_BT(\nabla_x + \nabla_y)P_s(\mathbf{x},\mathbf{y}) = 0. \tag{9}$$

The unique solution of (1) and (2) which is normalizable is $P_s(\mathbf{x},\mathbf{y}) \propto \delta(\mathbf{x}-\mathbf{y})\exp[-\beta V(\mathbf{x})]$, which corresponds to the two particles sitting on top of each other. In this case, $P(d) = \delta(d)$, i.e., $\lambda = 1$ [see Eq. (2)].

In conclusion, we have shown quite generally that nonlinear systems coupled through identical strong noises may have the same trajectory in the long time limit. In this context we were able to explain recent findings by Fahy and Hamann [10] on molecular dynamics studies of pairs of particles subjected to the same random forces, using Langevin dynamics. It is interesting to compare this new effect with the well known Casimir effect [13]. The latter case corresponds to the interaction between two parallel plates in vacuum due to quantum fluctuations of the electromagnetic field. It also manifests itself classically in the form of a long range interaction between two plates immersed in a system having critical fluctuations such as a binary fluid at its consolute point [14]. The effective attraction between trajectories mediated by the presence of the identical noise may be thought of as a generalized (dynamical) Casimir effect.

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[1] P. Berge, Y. Pomeau, and C. Vidal, *Order Within Chaos* (Wiley, New York, 1986).
[2] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[3] The quenching effect is observed when the x, y, or z equations or any combinations thereof are modified by a noise term—the key requirements seem to be that the noise be asymmetric and the *same* for both the trajectories.
[4] K. Matsumoto and I. Tsuda, J. Stat. Phys. **31**, 87 (1983).
[5] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).
[6] W. L. Ditto, S. N. Rauseo, and M. L. Spano, Phys. Rev. Lett. **65**, 3211 (1990).
[7] R. Georgii, W. Eberl, E. Luscher, and A. Hubler, Helv. Phys. Acta. **62**, 291 (1989).
[8] A. Hubler, Helv. Phys. Acta **62**, 343 (1989).
[9] We are indebted to Henry Abarbanel for this observation.
[10] S. Fahy and D. R. Hamann, Phys. Rev. Lett. **69**, 761 (1992).
[11] B. Derrida, in *Fundamental Problems in Statistical Mechanics VII*, edited by H. van Beijeren (North-Holland, Amsterdam, 1990), has studied, in a variety of contexts, the coalescence of trajectories with Langevin-type equations of systems with discrete degrees of freedom. Our arguments apply to this case too.
[12] See, e.g., C. W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 1985).
[13] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. **51**, 793 (1948).
[14] M. E. Fisher and P.-G. de Gennes, C. R. Acad. Sci. Ser. B **287**, 207 (1978).