Lattice Calculation of the Decays $B \to K^* + \gamma$ and $B_s \to \phi + \gamma$

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A lattice calculation of the form factors that determine the "hadronization ratios," such as R_{K^*} and R_ϕ, where $R_{K^*} \equiv [\Gamma(B \to K^* \gamma)/\Gamma(b \to s\gamma)]$, is presented in the quenched approximation. Lattice data shows strong evidence for the scaling law suggested by heavy quark symmetry for one of the form factors (i.e., T_2). The data also gives strong support for the simple pole ansatz for the q^2 dependence of T_2 in the range of available q^2. We thus find $T_2(0) = 0.10 \pm 0.01 \pm 0.03$, yielding $R_{K^*} = (6.0 \pm 1.2 \pm 3.4)\%$; we also find $R_\phi = (6.6 \pm 1.3 \pm 3.7)\%$.

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The loop decays of the b quark have long been noted for their capacity to provide important tests of the standard model (SM). Since many of these decays are short-distance dominated at the quark level, their inclusive rates are amenable to perturbation theory. Thus for inclusive processes reliable predictions can be made. The simple decay $b \to s + \gamma$ is a case in point. Recall that it is predicted [1] to have a branching ratio that varies from $\simeq 2 \times 10^{-4}$ to $\simeq 4 \times 10^{-4}$ as the top quark mass varies from 100 to 200 GeV. However, this inclusive process is challenging to measure experimentally, whereas a corresponding exclusive mode (i.e., $B \to K^* + \gamma$) has a distinctive signature and is much more accessible to experiment. Thus a meaningful confrontation between experiment and the underlying electroweak theory can be facilitated through a knowledge of the "hadronization ratio," R_{K^*}:

$$R_{K^*} \equiv \frac{\Gamma(B \to K^* \gamma)}{\Gamma(b \to s\gamma)}, \tag{1}$$

which is the probability for the formation of the K^*. The evaluation of this ratio by continuum methods has proven to be extremely difficult. This is reflected in the wide range $\sim 1\%$ to $\sim 97\%$ in the value of R_{K^*}, as calculated by quark models, QCD sum rules, heavy quark symmetry (HQS) extended to include the s quark, etc. (see Table I) [2]. It is thus clearly important to explore the use of lattice methods for treating such exclusive decays of B mesons.

At the quark level the decay is described by an effective Hamiltonian [1,2]:

$$H_{\text{eff}} = G_{g_1 g_2 g_3}(m_t, \mu) V_{ts} \bar{s}(x) \sigma_{\mu\nu} b_R(x) F^{\mu\nu}(x), \tag{2}$$

where $F^{\mu\nu}$ is the photon field strength tensor, and the c number coefficient $G_{g_1 g_2 g_3}(m_t, \mu)$ depends on all three gauge couplings of the standard model, the mass (m_t) of the top quark and a renormalization point μ. V_{tb} is assumed to be 1.

As usual the lattice is used for a nonperturbative evaluation of the matrix element $M_\mu \equiv \langle V(k)|J_\mu|P(p)\rangle$, where P is the initial pseudoscalar heavy-light meson, V is the final vector meson, $J^\mu \equiv \bar{s}\sigma^{\mu\nu} q_\nu b_R = (v+a)_\mu$ is the current, with v_μ and a_μ the vector and axial parts, and $q \equiv p - k$ is the 4-momentum of the photon. In general, the Euclidean matrix element can be parametrized in terms of three form factors [2,3]:

$$M_\mu = 2\epsilon_{\mu\nu\lambda\sigma}\eta^\nu(k) p^\lambda k^\sigma T_1(q^2) + [\eta_\mu(k)(m_H^2 - m_V^2) - \eta\cdot q(p+k)_\mu] T_2(q^2) + \eta\cdot q\left[q_\mu - \frac{q^2}{m_H^2 - m_V^2}(p+k)_\mu\right] T_3(q^2). \tag{3}$$

[Our γ matrices obey $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, and momenta are defined by $p_\mu = (E, i\mathbf{p})$.] For a lattice calculation it is simpler to note that the $T_1(q^2)$ term arises purely from the vector piece of J_μ and the T_2 and T_3 terms given above arise from the axial piece. The third term does not contribute when the photon is on shell. Furthermore, at the end point, $[q^2 = q^2_{\max} \equiv (m_H - m_V)^2]$, where the final and initial mesons are both at rest, T_3 term does not contribute to the axial matrix element. Since also at that kinematic point no momentum injection is required, $T_2(q^2_{\max})$ can be readily, and rather cleanly, evaluated on the lattice. Although $q^2 = 0$ (the point of direct physical interest) is not exactly accessible to the lattice, in many instances the parameters used in the current simulation do allow q^2 to be extremely small, i.e., $q^2/m_H^2 \lesssim 0.1$. Finally we note that using the identity $\sigma_{\mu\nu}\gamma_5 \equiv -\frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\sigma^{\lambda\rho}$ one can show that

$$T_2(0) = T_1(0). \tag{4}$$

Now the hadronization ratio of interest takes the simple form (for $m_s \ll m_b$):

$$R_{K^*} = 4\left(\frac{m_B}{m_b}\right)^3 \left[1 - \frac{m_K^{*2}}{m_B^2}\right]^3 |T_1(0)|^2. \tag{5}$$

On current lattices $q^2 = 0$ (or near that point) is inacces-

TABLE I. A sample compilation of the predictions for $R_{K^*} \equiv [\Gamma(B - K^*\gamma)/\Gamma(b \to s\gamma)]$. See Ref. [2].

Author(s)	R_{K^*}
O'Donnell (1986)	97%
Deshpande *et al.* (1988)	6%
Domingues *et al.* (1988)	$28 \pm 11\%$
Altomari (1988)	4.5%
Deshpande *et al.* (1989)	6–14%
Aliev *et al.* (1990)	39%
Ali *et al.* (1991)	28–40%
Du *et al.* (1992)	69%
Faustov *et al.* (1992)	6.5%
El-Hassan *et al.* (1992)	~0.7%–12%
O'Donnell *et al.* (1993)	~10%
Ali *et al.* (1993)	$13 \pm 3\%$
Ball (1994)	$20 \pm 6\%$
This work	$(6.0 \pm 1.2 \pm 3.4)\%$

sible for very heavy meson masses, say $m_H \gtrsim 3.5$ GeV. So at $m_H \sim m_B$, $T_1(0)$ is not directly calculable. However, HQS [4] allows one to predict the behavior of $T_2(q^2_{\rm max})$ at large m_H. Indeed, when $q^2 = q^2_{\rm max}$ no large momenta is transferred to the recoiling light hadron, so a straightforward argument shows that $\sqrt{m_H} T_2(q^2_{\rm max}) \to$ const (up to logarithms) as $m_H \to \infty$. This makes possible a controlled extrapolation of $T_2(q^2_{\rm max})$. Our strategy on the lattice will thus be as follows.

(1) Test pole dominance of T_2 at fixed m_H, to the extent that the data allow, by deducing $T_2(0)$ from $T_2(q^2_{\rm max})$ using the equation

$$T_2(0) = T_2(q^2_{\rm max}) \left[1 - \frac{q^2_{\rm max}}{m_H^2} \right] \qquad (6)$$

and comparing to $T_1(0)$ using Eq. (4). $T_1(0)$ is also obtained using pole dominance, but only from T_1 at small values of q^2 ($q^2/m_H^2 < 0.1$). Pole dominance does not appear to work well for $T_1(q^2)$ with large q^2.

(2) Extract $T_2(q^2_{\rm max})$ at $m_H = m_B$ by fitting the data to the form suggested by HQS, namely,

$$\sqrt{m_H} T_2(q^2_{\rm max}) = A_1 + A_2 \left(\frac{1}{m_H} \right). \qquad (7)$$

(3) Use pole dominance for T_2 at $m_H = m_B$ to deduce $T_1(0) = T_2(0)$ from $T_2(q^2_{\rm max})$.

We remark that in testing pole dominance, we have simply used the pseudoscalar mass in Eq. (6), as in the limit of large m_H, HQS implies that resonances of different spin parities become degenerate [4]. Note also that step (3) uses pole dominance over a wider range of q^2 ($q^2_{\rm max}/m_B^2 \approx 0.65$) than can be explicitly checked in step (1) ($q^2_{\rm max}/m_H^2 \lesssim 0.3$). We attempt to estimate below the systematic error associated with this step.

We mention the following technical points, regarding the lattice calculations, in brief [5].

(1) The recently proposed normalization of the Wilson quarks on the lattice [6–8]:

$$\psi^{\rm continuum} = \sqrt{2\tilde{\kappa} \exp(a\tilde{m})}\, \psi^{\rm lattice}, \qquad (8)$$

where

$$a\tilde{m} = \ln \left[\frac{1}{2\tilde{\kappa}} - 3 \right]$$

and $\tilde{\kappa} = \kappa/8\kappa_c$ (κ_c is the critical hopping parameter) is used. Thus the leading corrections that become important as am gets large are automatically included.

(2) For the renormalization of the tensor current we incorporate the correction calculated in lattice weak coupling perturbation theory to one loop order [9]. However, following Lepage and MacKenzie [7], the tadpole contribution is removed from the correction [it is already included in Eq. (8)], and a "boosted" value of $g_3 = g_V(1/a)$ is used.

We have done the calculation of T_1 and T_2 on four different sets of lattices: (A) $\beta = 6.3$, $24^3 \times 61$ (20 configurations, $a^{-1} = 3.01$ GeV); (B1) $\beta = 6.0$, $24^3 \times 39$ (8 configurations, $a^{-1} = 2.29$ GeV); (B2) $\beta = 6.0$, $24^3 \times 39$ (a second set of 8 configurations, $a^{-1} = 2.29$ GeV); and (C) $\beta = 6.0$, $16^3 \times 39$ (19 configurations, $a^{-1} = 2.10$ GeV). The lattice spacings given above are determined through a calculation of f_π with the same point sources that are used here [8]. The "B" is always taken at rest; the "K^*" is given lattice momentum (0,0,0), (1,0,0), (1,1,0) or (2,0,0), with (2,0,0) used only on B1 and B2. Preliminary results of this computation have been presented previously [10].

We first work in the case when the masses of the two light quarks are held equal. Experimentally this situation corresponds to the decay, for example, $B_s \to \phi + \gamma$. For the light quark we use $\kappa = 0.148$ at $\beta = 6.3$ and $\kappa = 0.152$ at $\beta = 6.0$, yielding a vector meson in the final state with mass ≈ 1.3 GeV. The dependence of the amplitude on the heavy quark mass is then studied. Specifically, for $\beta = 6.3$, we use $\kappa = 0.140$, 0.125, 0.110, and 0.100 for the heavy quark. Results are given in Table II; the last column shows that $\sqrt{m_H} T_2(q^2_{\rm max})$ is approximately constant. We then fit the data to the two parameter form [Eq. (6)] suggested by HQS, taking the correlations in the data into account through covariant fits. For the $\beta = 6.3$ data we find $A_1 = 0.806 \pm 0.069$ (GeV)$^{1/2}$, $A_2 = -0.545 \pm 0.082$ (GeV)$^{3/2}$, ($\chi^2/dof \approx 2.3/2$). Thus

$$T_2(m_H = m_{B_s}, q^2_{\rm max}) = 0.304 \pm 0.030.$$

We now discuss the systematic errors on T_2, first considering those relevant to $B_s \to \phi + \gamma$. To correct for the physical s quark we also study the matrix elements with $\kappa = 0.149$ (corresponding to vector meson of mass about 1.1 GeV), at $\beta = 6.3$. We find a shift in T_2, from its value at $\kappa = 0.148$ of about -7%. Extrapolating to the physical s quark would give a shift of -10%. In passing we mention that a similar study of our lattices at $\beta = 6.0$ indicates a smaller error than the 10% seen at $\beta = 6.3$.

We now assess the systematic error due to the use of

TABLE II. Lattice data on four sets of lattices. κ_1 represents the initial heavy quark undergoing weak decay, κ_2 the light quark emerging from the weak decay. The spectator quark is taken to have κ_2 as well. m_H and m_V are the masses of the initial and the final 0^- and 1^- mesons, respectively, and $r_{\max} \equiv [q^2_{\max}/m_H^2]$.

$\beta(a^{-1}/\text{GeV})$ $\{\kappa_c\}$	Lattice set	κ_1,κ_2	am_H	am_V	$r_{\max}$	$T_2(q^2_{\max})$	$\sqrt{m_H}T_2(q^2_{\max})$ (GeV$^{1/2}$)
6.3(3.01) {0.151}	A	140,148	0.590	0.422	0.081	0.406 ± 0.046	0.54 ± 0.06
	A	125,148	0.934	0.422	0.301	0.384 ± 0.044	0.64 ± 0.08
	A	110,148	1.248	0.422	0.443	0.364 ± 0.048	0.71 ± 0.10
	A	100,148	1.465	0.422	0.508	0.346 ± 0.055	0.73 ± 0.12
6.0(2.29) {0.157}	B1	135,152	0.894	0.561	0.139	0.409 ± 0.090	0.58 ± 0.13
	B1	118,152	1.244	0.561	0.301	0.371 ± 0.105	0.63 ± 0.18
6.0(2.29)	B2	135,152	0.891	0.566	0.139	0.478 ± 0.090	0.69 ± 0.13
	B2	118,152	1.245	0.566	0.301	0.415 ± 0.105	0.71 ± 0.17
6.0(2.10)	C	142,152	0.734	0.564	0.053	0.470 ± 0.062	0.58 ± 0.08
	C	135,152	0.888	0.564	0.133	0.459 ± 0.065	0.63 ± 0.09
	C	118,152	1.241	0.564	0.298	0.414 ± 0.089	0.67 ± 0.14

heavy quarks with $am \gtrsim 1$. For that purpose, we fit to the two parameter form using the two lightest quarks from our heavy set (of four) at $\beta=6.3$; i.e., we retain only $\kappa=0.140$ and $\kappa=0.125$. We find a shift in the value of T_2 of 3.1%.

To estimate scale breaking errors we compare the fit for the $\beta=6.3$ data with the heavy quarks at $\kappa=0.140$ and 0.125 to the fit for the $\beta=6.0$ data with the corresponding heavy quarks at $\kappa=0.135$ and 0.118. We attribute the difference of 12.2% to scaling violations.

The systematic errors due to finite size effects are deduced by comparing the value of T_2 on our 16^3 lattice with the one on the 24^3 lattice, both at $\beta=6.0$. We find a difference of 9.4%.

Adding in quadrature the errors due to the four sources mentioned above we find a total systematic error of 19%. In passing we note, however, that the systematic error due to each of these four sources is actually smaller than the statistical errors in the appropriate subset of data. It is, therefore, quite likely that the estimate of 19% is a conservative one. Thus, we arrive at

$$T_2(m_H=m_{B_s},q^2_{\max})=0.304\pm0.030\pm0.057\,. \quad (9)$$

Table III summarizes our test of the pole dominance for T_2. By examining the agreement between $T_2(0)$ and $T_1(0)$ we see that, within the available range of $q^2_{\max}/m_H^2 \le 0.3$, the pole model seems to work very well. We must note, however, that in the actual physical reactions of interest $q^2_{\max}/m_B^2$ approaches about 0.65. To estimate the error involved, we note that the biggest difference between $T_1(0)$ and $T_2(0)$ is ~9% (for lattice C). Scaling by the increased range in q^2 for the physical reaction (0.3 → 0.65), we arrive at an error of 20%. Since the data points with higher $q^2_{\max}/m_H^2$ in Table III seem to support pole ansatz just as well as those with lower values of $q^2_{\max}/m_H^2$, this is likely to be an overestimate, but we wish to be conservative. Using Eqs. (4), (5), (6), and (9) we thus find

$$T_1^{B_s\to\phi}(0)=T_2^{B_s\to\phi}(0)=0.104\pm0.010\pm0.028\,, \quad (10)$$

$$R_\phi=(6.6\pm1.3\pm3.7)\%\,, \quad (11)$$

which is the hadronization ratio for $B_s \to \phi+\gamma$. Note that in this calculation we have taken $m_b=4.5$ GeV, so that we may use the result for $B(b\to s\gamma)$ given by Misiak [1]. A 13% uncertainty is added in quadrature to

TABLE III. Test of the pole model for the q^2 dependence of the form factors, in particular, that of T_2. $T_2(0)$ and $T_1(0)$ are deduced, from $T_1(q^2)$ and $T_2(q^2)$ seen on the lattice, by using pole dominance, i.e., Eq. (6). Note $r \equiv q^2/m_H^2$.

Lattice set	κ_1,κ_2	r	$r_{\max}$	$T_1(q^2)$	$T_2(q^2_{\max})$	$T_1(0)$	$T_2(0)$
A	125,148	0.002	0.300	0.259 ± 0.035	0.384 ± 0.044	0.259 ± 0.035	0.269 ± 0.032
B1	135,152	0.009	0.139	0.391 ± 0.069	0.409 ± 0.090	0.388 ± 0.068	0.352 ± 0.077
B2	135,152	0.009	0.139	0.436 ± 0.092	0.478 ± 0.090	0.432 ± 0.091	0.411 ± 0.078
B1	118,152	−0.034	0.301	0.264 ± 0.050	0.371 ± 0.105	0.272 ± 0.051	0.260 ± 0.073
B2	118,152	−0.034	0.301	0.316 ± 0.110	0.415 ± 0.100	0.327 ± 0.113	0.290 ± 0.070
C	118,152	−0.069	0.298	0.300 ± 0.039	0.414 ± 0.089	0.321 ± 0.042	0.291 ± 0.062

the systematic errors on R corresponding to an assumed 200 MeV uncertainty in m_b.

Next we turn our attention to $B \to K^* + \gamma$. For this purpose we study matrix elements with unequal masses for the light quarks. Furthermore, we have to extrapolate in the masses of the two quarks. In particular, the spectator quark requires extrapolation to the chiral limit (i.e., $\kappa_c = 0.157$ at $\beta = 6.0$). For this study we use the $\beta = 6.0$, 24^3 lattice as it has the largest physical volume. This lattice has two independent sets of configurations with eight configurations in each sample. Consequently we arrive at

$$T_2(m_H = m_B, q^2_{\max}) = 0.325 \pm 0.033 \pm 0.065 . \quad (12)$$

Once again we use pole dominance to get

$$T_1^{B\to K^*}(0) = T_2^{B\to K^*}(0) = 0.101 \pm 0.010 \pm 0.028 , \quad (13)$$

$$R_{K^*} = (6.0 \pm 1.2 \pm 3.4)\% . \quad (14)$$

Now, as mentioned earlier, the inclusive branching ratio for $b \to s\gamma$ is predicted to lie in the range of about $(2\text{-}4) \times 10^{-4}$ depending on m_t. Thus, in the SM, there is a bound, $B(b \to s + \gamma) \leq 4 \times 10^{-4}$, corresponding to $m_t \approx 200$ GeV. Combining this upper bound with the above lattice result one gets

$$B(B \to K^* \gamma) \lesssim (2.4 \pm 0.5 \pm 1.4) \times 10^{-5} . \quad (15)$$

We recall now the recent CLEO result [11]:

$$B(B \to K^* \gamma) = (4.5 \pm 1.5 \pm 0.9) \times 10^{-5} . \quad (16)$$

Given the size of the errors in the lattice calculation, as well as in the experiment, the CLEO result is certainly *not inconsistent* with the expectations based on the lattice. However, we note that the numbers seem to mildly favor a rather heavy top quark.

In an attempt to quantify the statement about m_t we note that the experimental result (16) along with the lattice result (14) imply

$$B(b \to s\gamma) \approx (7.5 \pm 5.4) \times 10^{-4} ,$$

where we have assigned a $\sim 70\%$ combined error to the lattice plus the experimental result. At the 1σ level one then finds $m_t \gtrsim 100$ GeV. However, modest improvements in the lattice and/or experimental results could produce a rather stringent bound.

To summarize, we have used lattice methods along with the HQS of QCD [4] to evaluate the form factors for the radiative B transitions. We also want to highlight two drawbacks of the present effort. First, numerical limitations did not allow us to check pole dominance for the specific value of the momentum transfer relevant to the experiment. We have included what we believe is a conservative estimate of 20% systematic error due to the use of pole dominance. The second limitation is, of course, the quenched approximation. It is generally believed that with the use of a physical quantity (e.g., f_π in our work [8]) to set the scale for the lattice calculations, errors due to quenching are likely to be quite small, perhaps $\leq 10\%$, in the form factors of interest here. It is, therefore, unlikely that the present limitations would seriously affect our results, given the $\sim 28\%$ error in amplitude. Quenched simulations are now in progress that should allow us to improve the calculations to the (10–15)% level. At that stage errors due to quenching may also start to become relevant.

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