

Fully Packed Loop Model on the Honeycomb Lattice

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We investigate the $O(n)$ model on the honeycomb lattice, using its loop representation in the limit of full packing. The universal properties, which we calculate by means of finite-size scaling and transfer-matrix techniques, are different from the branches of $O(n)$ critical behavior known thus far. The conformal anomaly of the model varies between -1 and 2 in the interval $0 \leq n \leq 2$. The universality class of the model is characterized as a superposition of a low-temperature $O(n)$ phase, and a solid-on-solid model at a temperature independent of n .

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The $O(n)$ model consists of n -component spins $\mathbf{s} = (s_1, s_2, \dots, s_n)$ on a lattice, with isotropic couplings [1]. Thus, the Boltzmann weight associated with the interaction between a pair (\mathbf{s}, \mathbf{t}) depends only on the scalar product of the spins: it has the form $w(\mathbf{s} \cdot \mathbf{t})$. According to the assumption of universality, the critical behavior of this model does not depend on the precise choice of the function w , at least not within reasonable limits. It is very fortunate that the possibility arises to choose $w(\mathbf{s} \cdot \mathbf{t}) = 1 + x\mathbf{s} \cdot \mathbf{t}$ where x plays the role of the inverse $O(n)$ temperature. As a consequence of this choice, the $O(n)$ partition integral takes the form

$$Z_{O(n)} = \int \cdots \int \left(\prod_k d^{n-1} \mathbf{s}_k \right) \prod_{\langle i,j \rangle} (1 + x\mathbf{s}_i \cdot \mathbf{s}_j), \quad (1)$$

where each spin is integrated over the surface of an n -dimensional sphere with radius \sqrt{n} . The second product is over all pairs of nearest neighbors. Equation (1) can be rewritten [2] as the partition sum of a loop model,

$$Z_{O(n)} = Z_{\text{loop}} = \sum_{\text{all } \mathcal{G}} x^{N_b} n^{N_l}, \quad (2)$$

where \mathcal{G} is a graph covering N_b bonds of the honeycomb lattice, and consisting of N_l closed, nonintersecting loops. Thus the variables n and x play the role of fugacity of the loops and covered bonds, respectively. The $O(n)$ model has positive definite Boltzmann weights only for $|x| < 1/n$, and ferro- or antiferromagnetic interactions for $x > 0$ or $x < 0$, respectively. In contrast, the loop model is physical for all positive x . Our present knowledge of the $O(n)$ model in two dimensions is largely based on this mapping and its analog for the square lattice [3-8].

A third interpretation of the partition sum is that of a triangular n -component corner-cubic model, in which each spin component independently takes one of the values ± 1 . Neighboring spins may be different in at most one component, so that interfaces between regions of different spins appear as loops on the honeycomb lattice.

Bonds between equal spins and those between different spins carry local Boltzmann weights 1 and x , respectively.

The weak coupling (high-temperature) limit of the $O(n)$ model corresponds to the low density regime of the loop model, and the strong coupling (low-temperature) limit of the cubic model. High density in the loop model corresponds to antiferromagnetism in the cubic model and the unphysical regime of the $O(n)$ model.

Whereas the degrees of freedom of the loop model (i.e., bonds are absent or present in \mathcal{G}) are discrete, the spin dimensionality n appearing in Eq. (2) can be considered a continuous variable. For the special choice [3]

$$x^{-2} = 2 \pm \sqrt{2-n}, \quad (3)$$

the model is exactly solvable [4-6]. The solution has two branches corresponding with the two signs in Eq. (3). These branches play the following role in the inferred phase diagram [3,9] shown in Fig. 1. Branch 1 (with $x^{-2} \geq 2$) is the $O(n)$ critical line separating the disordered phase at small x from the spin-ordered phase at large x . Thus, under renormalization, branch 1 acts as a line of unstable fixed points. Branch 2 (with $x^{-2} \leq 2$) lies in the ordered phase and plays the role of a line of stable fixed points [9] describing that phase, commonly known as the low-temperature phase of the $O(n)$ model. According to Eq. (2), it is equivalent to a system of densely (but not fully) packed loops. Thus we refer to branch 2 as the DPL (densely packed loop) model. The deduced renormalization flow is indicated in Fig. 1 for one value of n .

Since each loop covers an even number of edges of the honeycomb lattice, the sign of x is redundant, and the phase diagram is symmetric with respect to the line $x^{-1} = 0$. This symmetry forbids that points with $x^{-1} = 0$ belong to the domain of attraction of the low-temperature $O(n)$ fixed line with $x^{-1} > 0$, or that with $x^{-1} < 0$. Therefore, it seems plausible that the line $x^{-1} = 0$ plays the role of an unstable fixed line, and we may expect new universal behavior. For this value of x^{-1} , only those configurations contribute to Eq. (2) in

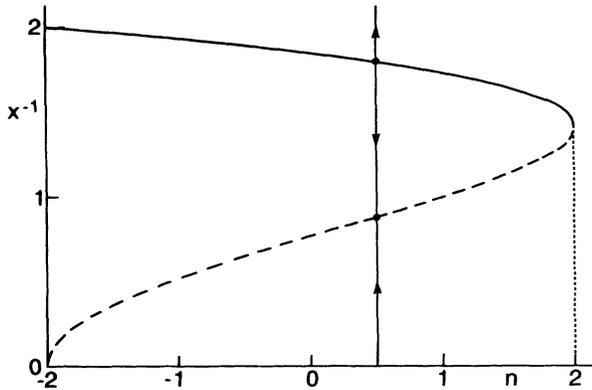


FIG. 1. Phase diagram of the $O(n)$ model on the honeycomb lattice as a function of n and the relative weight x^{-1} of an empty vertex. The model is exactly solvable on the curve shown. The full part (branch 1) separates the disordered $O(n)$ phase at large x^{-1} from the low-temperature phase at small x^{-1} . The dashed part of the curve (the DPL model) is interpreted as a line of stable fixed points. Since the model is symmetric with respect to the sign of x , the line $x^{-1} = 0$ (the FPL model) acts as an unstable fixed line.

which all vertices are visited by a loop. Since the coordination number is 3, only two-thirds of all lattice edges are covered by a loop. However, since the lattice cannot accommodate a higher density, we refer to this line by the fully packed loop (FPL) model. We note that there is no direct relationship between the honeycomb FPL model and the Potts model (such a relation does exist for other lattices [10]).

The FPL model is the subject of our investigation. A first question is whether its universal behavior corresponds to known classes. Besides the branches 1 and 2 mentioned above, three other branches have been found in the square lattice $O(n)$ model [7,8]. One of these is called "branch 0" and is exactly equivalent [7] with a critical q -state Potts model with $\sqrt{q} = n + 1$. This relation is *not* the one due to Temperley and Lieb [10] which reads $\sqrt{q} = n$ and applies, via arguments of universality, to branch 2. Branch 3 describes a higher order critical point where an $O(n)$ and an Ising critical line merge into a first-order line [7]. Branch 4 can be interpreted as a superposition of a critical Ising model and a low-temperature $O(n)$ model.

To investigate the universal properties of the FPL model, we have used the transfer-matrix technique described in Ref. [9]. The model is wrapped on an infinitely long cylinder, with the axis parallel to one of the three edge directions. The finite-size parameter L is defined as the number of elementary hexagons spanning the cylinder, which defines the unit of length as $\sqrt{3}$ times a lattice edge. The transfer matrix adding one layer of hexagons is denoted \mathbf{T} . Its largest eigenvalue λ_0 determines the free energy f_L per unit of area,

$$f_L = \frac{2}{L\sqrt{3}} \ln \lambda_0, \quad (4)$$

and the next-largest eigenvalues λ_i ($i = 1, 2, \dots$) determine correlation lengths $\xi_{L,i}$ according to

$$\xi_{L,i}^{-1} = \frac{2}{L\sqrt{3}} \ln \frac{\lambda_0}{\lambda_i}, \quad (5)$$

so that the associated correlation functions $g_{L,i}(r)$ over a distance r along the cylinder behave asymptotically as

$$g_{L,i}(r) \sim e^{-r/\xi_{L,i}}. \quad (6)$$

The theory of conformal invariance predicts that [11,12]

$$f_L \simeq f_\infty + \frac{\pi c}{6L^2}, \quad (7)$$

where c is the conformal anomaly [13], and that the scaled gaps satisfy [14]

$$L\xi_{L,i}^{-1} \simeq 2\pi X_i, \quad (8)$$

where X_i is the scaling dimension of the observable correlated by $g_{L,i}$.

Thus we have determined the conformal anomaly c , and two scaling dimensions X_t and X_h from numerically calculated eigenvalues of \mathbf{T} for values of L up to 15. The dimension X_t is determined by the two largest eigenvalues of \mathbf{T} associated with translationally invariant eigenstates, in analogy with the determination of the temperature exponent of $O(n)$ models at finite temperatures [5,7]. However, the second-largest eigenvalue for the FPL model is *not* the continuation of that for finite-temperature $O(n)$ models: intersection of eigenvalues occurs at packing densities above that of branch 2. It is natural to associate X_t with the temperaturelike variable x^{-1} .

The magnetic dimension X_h was determined from the ratio of the leading eigenvalue of \mathbf{T} and that of a modified transfer matrix obtained by adding one loop segment running in the length direction of the cylinder [7]. This segment can be closed into a loop by closing the cylinder into a torus. The eigenvalue ratio thus obtained for the FPL model is the analytic continuation of that used for the finite-temperature $O(n)$ model [9].

Since the finite-size results display oscillations as a function of L with period 3, only multiples of 3 were used in the analysis. The transfer matrix of the FPL model contains a conserved quantity: the number of nonintersecting strings as described in Ref. [15]. This leads to a factorization of \mathbf{T} in invariant sectors. Apart from this factorization, the calculations were analogous to those of Ref. [9].

Furthermore, the results are restricted to $n \geq 0$ because of an intersection of eigenvalues: a different (apparently unphysical) eigenvalue of \mathbf{T} dominates the spectrum for negative n . The extrapolation procedures lead-

ing to our best estimates shown in Table I are described, e.g., in Ref. [16].

It is obvious from these data that the fully packed loop model behaves differently from the five known branches of $O(n)$ critical points. Our results do, however, reveal a relation with the low-temperature $O(n)$ model and the equivalent DPL model:

$$\begin{aligned} c^{\text{FPL}} &= c^{\text{DPL}} + 1 = 2 - 6(1 - g)^2/g, \\ X_h^{\text{FPL}} &= X_{0,1}^{\text{DPL}} = 1 - 1/(2g), \\ X_t^{\text{FPL}} &= X_{1,2}^{\text{DPL}} + 1 = 3g/2, \end{aligned} \tag{9}$$

where $g = (1/\pi) \arccos(-n/2)$. The pairs of subscripts apply to the conformal classification of scalar operators (i.e., $X_{p,q} = 2\Delta_{p,q}$ [17]). In the DPL model the exponent $X_{0,1}$ governs the asymptotic behavior of the probability that two lattice points lie on the same loop, as can be deduced from a mapping on the Coulomb gas [18]. Thus, the corresponding transfer-matrix eigenvalue is expected in the sector characterized by *two* loops spanning the long dimension of the torus. Indeed, numerical results confirm this, for the DPL as well as for the FPL model. In the latter model, the largest eigenvalues in the sectors with one and with two loops in the length direction of the torus appear to be identical.

For comparison, the relevant quantities of the DPL model are included in Table I. The data agree well with the identification made above except for some small deviations at $n = 2$. These can be satisfactorily explained by slowly converging logarithmic corrections at the Kosterlitz-Thouless critical state [19], which occurs both at the point where branches 1 and 2 meet and at lower $O(n)$ temperatures at $n = 2$ [9].

Thus the FPL model displays simultaneously the universal properties of a low-temperature $O(n)$ or DPL model and those of a model with $c = 1$ and $X_t = 1$.

TABLE I. Numerical results for the conformal anomaly c , the magnetic dimension X_h , and the temperature dimension X_t of the fully packed loop (FPL) model. For comparison, we include the conformal anomaly and the scaling dimensions $X_{0,1} = 2\Delta_{0,1}$ and $X_{1,2} = 2\Delta_{1,2}$ (using the notation of Ref. [17]) of the low-temperature $O(n)$ model and the equivalent densely packed loop (DPL) model.

n	c^{FPL}	X_h^{FPL}	X_t^{FPL}
0.0	-1.000 (0)	0.0 (0)	0.750 (5)
0.5	0.180 (2)	0.1386 (1)	0.86 (2)
1.0	1.000 (5)	0.2500 (1)	1.00 (1)
1.5	1.59 (1)	0.350 (1)	1.16 (1)
2.0	2.00 (1)	0.470 (1)	1.46 (1)
n	c^{DPL}	$X_{0,1}^{\text{DPL}}$	$X_{1,2}^{\text{DPL}}$
0.0	-2	0	-0.25
0.5	-0.820	0.1386	-0.129
1.0	0	0.25	0
1.5	0.588	0.3506	0.155
2.0	1	0.5	0.5

This observation is in line with an exact mapping of the $n = 1$ FPL model on a solid-on-solid (SOS) model. This mapping [15] is summarized as follows. (i) Remove those edges of the dual (triangular) lattice that do not intersect a loop (see Fig. 2). This leads to a “diamond” tiling of the plane. (ii) Interpret the tiling three dimensionally, i.e., as a stack of cubes, or a crystal surface projected in the (1,1,1) direction. For $n = 1$, the loop weights are unity and therefore the SOS weights are strictly local.

This SOS model has $c = 1$ [20]. It is also equivalent with the zero-temperature antiferromagnetic Ising model on the *triangular* lattice [15] which has a temperature dimension $X_t = 1$ [21]. These two exact results coincide with the corresponding entries in Table I, in agreement with the interpretation of the FPL model as a superposition of a low-temperature $O(n)$ (or DPL) model and an SOS model. The low-temperature $O(1)$ model (the point on branch 2 at $n = 1$) does not contribute to c and X_t : it is a frozen, zero-temperature ferromagnetic Ising model on the honeycomb lattice. Furthermore, for $n = 1$ the variable x^{-1} corresponds with the Ising temperature in the zero-temperature antiferromagnetic Ising model on the triangular lattice, in agreement with our earlier assumption that X_t is associated with x^{-1} .

For $n \neq 1$ the configurations of the fully packed loop model may still be identified as the configurations of the SOS model, but now the weights have a nonlocal component due to the fugacity of the loops. The model can be viewed as having both loops and SOS variables in intimate relation. From the numerical results for the central charge and the exponents we are led to the remarkable conclusion that the interaction between the two types of variables is irrelevant, so that the critical behavior is that of the superposition of the two models. Since SOS exponents are temperature dependent, and the SOS contribution to X_t^{FPL} is constant, the renormalized temperature of the SOS model does not depend on n .

This interpretation of the FPL model as a superposition of two models is analogous to that of branch 4 of

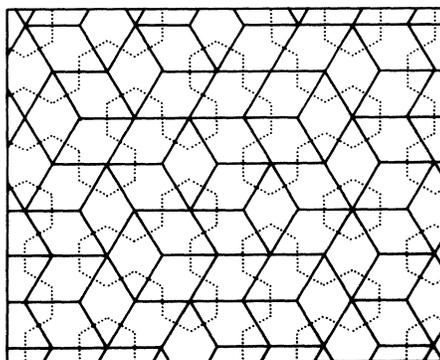


FIG. 2. Relation between the degrees of freedom of the FPL model on the honeycomb lattice and those of an SOS model on the triangular lattice.

the $O(n)$ model on the square lattice [7]. In that case the loop model configurations contain Ising-like degrees of freedom, and the universality class is that of decoupled Ising and low-temperature $O(n)$ models, even though both models are interwoven on a microscopic level.

The conformal anomaly reaches a value as high as 2 for $n \rightarrow 2$. High values of c have earlier been reported in models describing the fully frustrated XY (FFXY) model [22,23]. This invites a comparison between the FFXY and the $n = 2$ FPL model. Frustration in the FFXY model is introduced by changing the sign of precisely one spin-spin interaction in each elementary face. In the language of the loop model [Eq. (2)] this corresponds with one negative bond weight in each elementary face. In this sense, frustration is absent in the FPL model. However, the zero-temperature triangular antiferromagnetic Ising model, which is dual to the $n = 1$ FPL model, is fully frustrated. This frustration is reflected in the present FPL model by the fact that a graph \mathcal{G} can cover only two-thirds of the edges of the honeycomb lattice. This suggests the possibility of frustrationlike phenomena in other loop models, for instance on the square lattice, when the vertex weights are chosen such that not all edges can be covered by \mathcal{G} .

Finally we discuss the large- n behavior of the FPL model. The numerical evidence concerning the finite-size scaling behavior of the magnetic gap shows that the model is not critical for $n \gg 2$. The limit $n \rightarrow \infty$ maximizes the number of loops and thus leads to a close packing of hexagons, resembling the ordered state of the hard-hexagon model. Local excitations appear in the form of loops that are somewhat larger than the elementary faces and do not destroy the long-range ordered state. In SOS language, this is a flat phase. Thus a roughening transition takes place in the FPL model when n varies. Our numerical data indicate that this transition is located at $n < 3$, and are consistent with the natural value $n = 2$.

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