Decay of Kadomtsev-Petviashvili Solitons

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The Kadomtsev-Petviashvili equation with positive dispersion has well known flat as well as two dimensional soliton solutions. The two dimensional solitons, however, involve regions of depressed as well as of excess density. It might therefore seem to be an open question if flat solitons, known to be unstable, would decompose into these rather strange structures. The present paper definitely answers this question in the affirmative by describing a numerical simulation.

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The physically interesting Kadomtsev-Petviashvili (KP) equation, considered in our paper, is unique among nonlinear partial differential equations in x,y,t space known to us, in that two distinct and very different classes of exact N soliton solutions exist. The first class comprises N ramplike solitons, crossing at various points and propagating each with its own velocity $\mathbf{v}_i$. The second kind are bounded in all directions and collide with each other without distortion or even change of phase. This dualism is remarkable in itself and is the cause of considerable interest to applied mathematicians, so much so that it took people some time to get used to. An important question, unanswered until now, was whether the second class was in fact more than just a mathematical artifact. The bounded solitons further differ from most of their unbounded brethren in that they are given mathematically by ratios of polynomials, rather than falling off exponentially, as the flat solitons for KP do.

A priori it might seem possible that the ramplike solitons could break up into an array of localized specimens. However, this ultimate fate is by no means obvious, as the latter kind has thicker "tails" than the former and these must be filled out.

In any case, the possible decay of one kind of exact N soliton solution into another is an important, tasty, and until now, unsolved problem of applied mathematics. The present paper formulates the problem in mathematical terms and solves it numerically for one initial soliton. It is seen that the decay products are in fact described by exact two dimensional solutions. The match will be seen to be almost perfect.

The Kadomtsev-Petviashvili equation [1]

$$(n_t+6nn_x+n_{xxx})_x+3\varepsilon n_{yy}=0,\quad \varepsilon=\pm 1\,, \tag{1}$$

describes the dynamics of two dimensional, small, but finite amplitude waves and solitons in a variety of media. Examples are the following: plasma physics, hydrodynamics, and solid state physics [2–4].

Equation (1) is derived in various physical contexts assuming that the soliton (or wave) is moving along x. All changes in y are slower than in the direction of motion. For an extensive derivation, see Ref. [5]. Equation (1) is integrable by inverse scattering [6].

Both variants of (1) are solved by the one soliton, ramplike solution,

$$n(x,t)=(\eta/2)\,\mathrm{sech}^2[\eta^{1/2}(x-\eta t)/2]\,, \tag{2}$$

where η is a parameter. This solution was investigated for stability with respect to two dimensional perturbations in a numerical simulation using (1) with $\varepsilon=+1$ (negative dispersion) [7]. It was found to be stable.

The same evolution problem is infinitely more interesting for $\varepsilon=-1$ (positive dispersion). First of all, the soliton (2) is known to be unstable with respect to linear perpendicular perturbations [1,8]. Second, a rather strange two dimensional, localized stationary soliton structure is known (as well as multisoliton solutions) [9]. It is

$$n=\frac{4v[1-v(x-3vt)^2+v^2y^2]}{[1+v(x-3vt)^2+v^2y^2]^2}\,. \tag{3}$$

Note that n, the excess of the density over the mean, can be negative (valleys appear around the x axis). The structure is stable in two dimensions, at least according to a limited stability analysis [10]. Thus one might wonder if (and how) the ramplike soliton (2) will break up into an array of localized solitons, possibly given by (3). Alternatively, it could simply disintegrate.

The Kadomtsev-Petviashvili equation is in some sense peculiar in that the initial condition must fulfill an infinite set of constraints if the solution is to remain localized in x [11]. The first of these constraints is obtained by integrating (1) over x:

$$\partial_{yy}\int_{-\infty}^{\infty} n\,dx=0\,. \tag{4}$$

Just adding a perturbation to the flat soliton solution (2)

$$n_{\text{perturbed}}=[1+\delta\cos(k_y y)]\,n_{\text{exact}} \tag{5}$$

would violate this constraint. Thus bending [adding, say, $\delta\cos(k_y y)$ to the *phase*] is a natural and consistent perturbation. This was the choice of the initial condition used, as in Ref. [7].

The results are presented in Fig. 1. The initial bending of the soliton produces a corresponding variation in the

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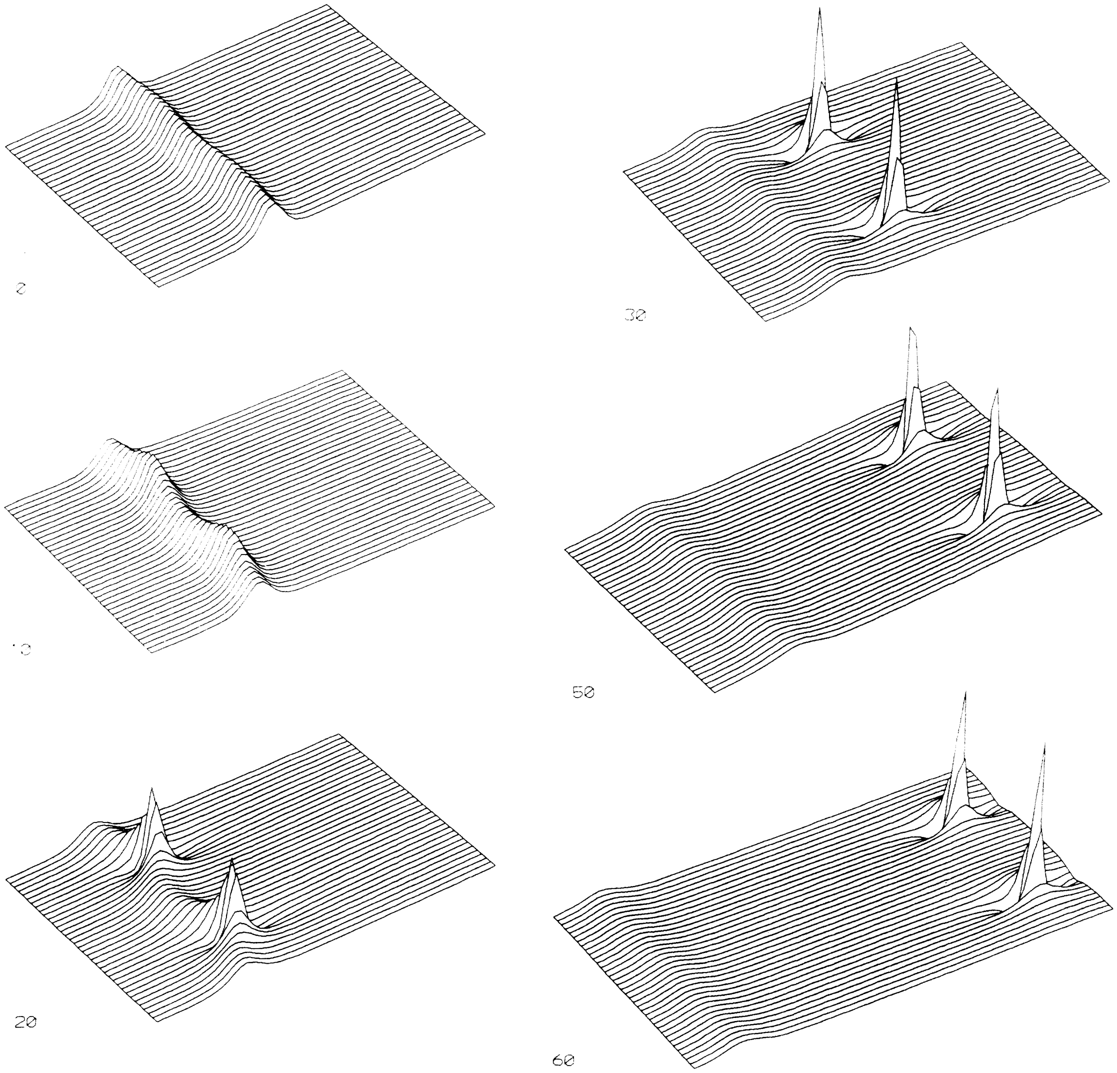

FIG. 1. Consecutive stages in transition from flat soliton to two dimensional soliton array, $n(x,t) \to n(x,y,t)$.

height (same k_y, chosen to correspond to the maximum growth rate of the instability). This is followed by the production of an array of two dimensional structures. They initially vary somewhat ($\sim 10\%$), but after a while become more robust and very well describable by (3) when v is adjusted. Figure 2 gives a graphic illustration of the soliton given by (3) with v adjusted so that the velocity is the same as that of the decay product in the last frame of Fig. 1. There can be no doubt that the decay product tends to a two dimensional soliton given by (3).

Comparison of our Fig. 1 with Fig. 1 of Ref. [7] reveals just how much more interesting this case is as compared with that of $\varepsilon = +1$.

When solving (1) numerically, the x and y derivatives were calculated by the fast Fourier transform method. The time evolution was calculated by the leapfrog scheme, which was shown to be numerically stable for sufficiently short time steps. Figure 2 was produced on the same numerical mesh as the results of the simulation (Fig. 1).

The next step would be a fully three dimensional simulation to see if our two dimensional structures will once again disintegrate. This sequence was followed in three dimensions for a simpler equation (Zakharov-Kuznetsov) in Ref. [12]. It is envisaged for (1) in the future.

Recently new, exact two dimensional solutions to (1) with $\varepsilon = -1$ were found [13,14]. One in particular, Fig. 1 of Ref. [13], somewhat resembles our fragments, but only at $t = 0$. After a while the similarity vanishes. All in all, formula (3) is the right one for our two dimensional solitons.

In conclusion, we see that a suitably tailored perturba-

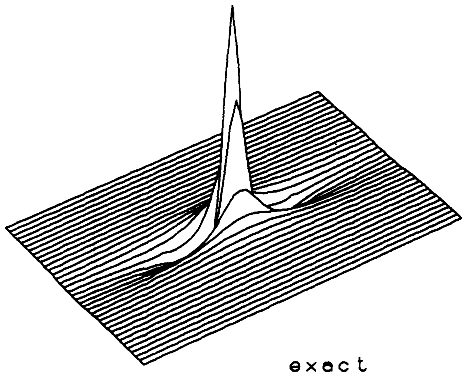

FIG. 2. Graphic representation of a soliton given by (3) with v adjusted so that the velocity of propagation from left to right is the same as in the last frame of Fig. 1.

tion of an unstable flat soliton will lead to decay into two dimensional, stationary soliton structures. This will happen in spite of the unusual character of these structures. Furthermore, they are seen to be two dimensionally stable, at least on the time scale of our simulation. New questions have been raised and must await a three dimensional simulation.

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