Observation of Induced Subcritical Bifurcation by Near-Resonant Perturbations

S. T. Vohra,[1] L. Fabiny,[1] and K. Wiesenfeld[2]
[1]Naval Research Laboratory, Code 5670, Washington, D.C. 203785
[2]Department of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332
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We present the first experimental evidence that near-resonant perturbations produce a destabilizing shift of a subcritical bifurcation. This is in direct contrast to supercritical bifurcations, where near-resonant perturbations always suppress the instability. Using a normal form analysis we derive a generic scaling law relating the magnitude of the destabilizing shift μ to the perturbation amplitude ε and detuning frequency δ. The predictions are in excellent agreement with our experiments. In the limit of very small δ the shift obeys a pure scaling law, $\mu \propto (\varepsilon)^{2/3}$.

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It is well known that any time-periodic nonlinear system is sensitive to near-resonant perturbations [1] when tuned near its bifurcation point. Moreover, since the essential dynamics reduce dramatically near a bifurcation, the system response obeys universal scaling laws. This has been demonstrated in a number of experiments including those on electrical circuits [2], NMR lasers [3], charge-density waves [4], spin waves [5], bouncing balls [6], Josephson junctions [7], and magnetostrictive oscillators [8]. For sufficiently small perturbations, linearized theories accurately predict the observed scaling behavior.

More recently, attention has turned to intrinsically nonlinear effects which arise for somewhat larger near-resonant perturbations. One such effect is the stabilizing shift of the bifurcation point: Near the onset of a supercritical bifurcation, periodic perturbations always act to suppress the instability. This effect has been observed in experiments on analog circuits [9], Josephson junctions [7], and magnetostrictive ribbons [8]. Theoretically, a normal form analysis of supercritical bifurcation has been successful in predicting (among other things) the scaling law for the shift as a function of the perturbation amplitude and frequency [9]. It was suggested that the opposite effect should occur for a system tuned near a subcritical bifurcation, i.e., that the instability would be *induced* rather than suppressed, though no quantitative analysis of the subcritical case exists.

In this Letter, we present clear experimental evidence that near-resonant perturbations produce a *destabilizing shift* near a subcritical period doubling bifurcation. Our measurements are performed on an as-cast (i.e., unannealed) magnetostrictive ribbon in a time-dependent magnetic field. While this system is known to display a rich variety of nonlinear dynamical behavior [10], no accurate theory for the dynamics of the ribbon is available; nevertheless, close to the bifurcation point we can make quantitative predictions by using the appropriate normal form which depends only on the bifurcation "type." In our case, we use the normal form appropriate to a subcritical period doubling bifurcation, from which we derive a closed form expression for the size of the shift as a function of the perturbation frequency and amplitude.

While our focus here is to study generic nonlinear effects, the problem of near-resonant perturbations is especially relevant to this system since the dynamic magnetostrictive response has been successfully exploited to detect very small magnetic fields [11]. We detect the nonlinear dynamical strain response of the magnetostrictive oscillator with a fiber optic (FO) interferometer capable of resolving dynamic ($f > 1$ kHz) strain on the order of 10^{-12}. It is thus possible to make extremely accurate strain measurements. Dynamic strain response of a magnetorestrictive oscillator has been used to study a variety of nonlinear phenomena including stochastic resonance [12], the (supercritical) noise rise near the bifurcation point and period doubling suppression [8] and control of chaos [13].

The experiment involved measuring the dynamic strain response of a magnetically driven $Fe_{78}B_{13}S_9$ amorphous magnetostrictive ribbon (Metglas 2605S-2) using a fiber-optic Mach-Zehnder interferometer. A small portion (< 5 mm) of the ribbon ($50\ \text{mm} \times 12\ \text{mm} \times 25\ \mu\text{m}$) was bonded to the optic fiber comprising one arm of the interferometer. The phase shift of light propagating in the fiber attached to the ribbon is a direct measure of strain in the ribbon. The interferometer was contained in a solenoid which was driven by a two channel frequency synthesizer (HP 3326A), providing a longitudinal magnetic field $H = H_{dc} + h_0 \cos 2\pi f_0 t$, where H_{dc} is the applied dc field and h_0 is the amplitude of the sinusoidally varying pump field. The perturbing signal, $h_1 \cos 2\pi f_1 t$, where h_1 and f_1 are the amplitude and the frequency, respectively, of the perturbing signal, was added with the second channel of the synthesizer. The power spectral density output of the strain response was measured with a dynamic signal analyzer (HP 3562A) and the amplitudes of the applied magnetic fields were obtained by measuring the voltage drop across a 1 Ω resistor in series with the solenoid. The experimental arrangement has been described in previous work [8,10,12].

Figure 1 shows clearly that a near-resonant perturbation induces the onset of a subcritical bifurcation. For the unperturbed case ($h_1 = 0$) the magnetostrictive oscillator displayed a distinct period doubling bifurcation for

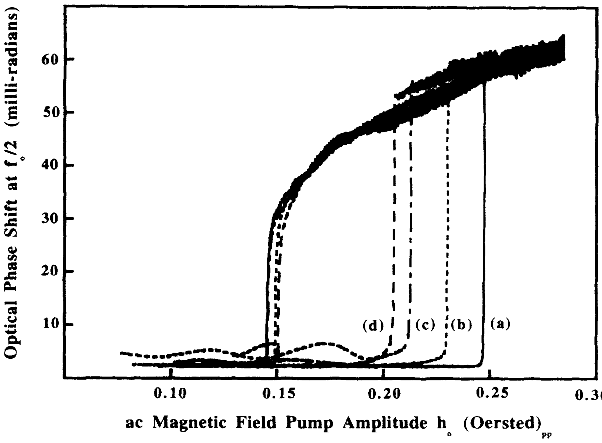

FIG. 1. Experimental data showing induced subcritical bifurcation by near-resonant perturbation: (a) subcritical period doubling bifurcation curve corresponding to the unperturbed bifurcation case ($h_1=0$), (b) $h_1=1$ mG, (c) $h_1=2.7$ mG, and (d) $h_1=6.8$ mG. The fixed experimental parameters were $f_0=9.55$ kHz, $H_{dc}=1.56$ G, and $\Delta=0.1$ Hz (corresponding to $\delta=10^{-5}$). The data were obtained with a fiber optic interferometer from the nonlinear strain dynamical response of a driven magnetostrictive oscillator.

$h_0=0.25$ Oe with $f_0=9.55$ kHz and $H_{dc}=1.56$ Oe. The strain response at $f_0/2$ was detected with a lock-in amplifier (LIA) and plotted as a function of the bifurcation parameter (h_0). The introduction of a perturbing periodic signal ($h_1>0$) was found to shift the bifurcation point in such a way as to *induce* the subcritical bifurcation. The bifurcation shift was expected and found to be a function of the perturbing signal strength h_1 and detuning, $\Delta=|f_0/2-f_1|$. For $\Delta=0.1$ Hz, Fig. 1 shows the measured strain response at $f_0/2$ as a function of the bifurcation parameter h_0 for various values of h_1. The behavior is in direct contrast with the case of supercritical bifurcations: Near-resonant perturbations *induce* subcritical bifurcations and *suppress* supercritical bifurcations.

In order to understand the effects of periodic perturbation on a subcritical bifurcation we use the augmented normal form [9],

$$\dot{x}=\mu x+x^3+\varepsilon\cos(\delta t)\,, \tag{1}$$

where $\varepsilon=0$ represents the unperturbed system of Eq. (1) and ε is proportional to the strength of the near-resonant perturbation (h_1) and δ is proportional to the detuning frequency. This is really a continuous time approximation of a one-dimensional map, which in the unperturbed case ($\varepsilon=0$) rigorously described the dynamics near the bifurcation point [14]. Augmenting the normal form with the periodic term is on less firm footing, though it can be justified within the context of perturbation theory [6]. The key point is that the period doubling bifurcation of the full nonlinear system coincides with a symmetry breaking bifurcation of Eq. (1). Our goal is to find an approximate analytic expression describing the shifted bifurcation point in terms of ε and δ from the augmented normal form [Eq. (1)]. The first step is to find the symmetric periodic solution $x_0(t)$ of Eq. (1). Using the method of harmonic balance [15], we take

$$x_0(t)=B\cos(\delta t+\phi)+\cdots\,,$$

where the ellipses represent higher harmonics. Substituting this into Eq. (1), ignoring the higher harmonics, and balancing the fundamental Fourier terms yield a cubic equation for B^2

$$B^2[\delta^2+(\mu+\tfrac{3}{4}B^2)^2]=\varepsilon^2\,. \tag{2}$$

The shifted bifurcation point can be obtained by finding the point where the symmetry breaking bifurcation occurs. Linearizing the normal form about the periodic solution $x_0=x-\eta$, one has immediately

$$\dot{\eta}=[\mu+3x_0^2(t)]\eta\,.$$

The perturbation η grows exponentially if the time average of $\mu+3x_0^2(t)$ is positive: η decays if it is negative. From our approximation for $x_0(t)$, the time average of $[x_0(t)]^2$ is $B^2/2$ so that

$$\mu_*+\tfrac{3}{2}B^2=0\,.$$

Combining this with Eq. (2) yields the desired result

$$\mu_*^3+4\delta^2\mu_*+6\varepsilon^2=0\,. \tag{3}$$

Since Eq. (3) is a cubic equation it is possible to find a closed form solution for μ_* in terms of δ and ε. Since all coefficients are positive, there can only be one real root and it will be *negative*. This means that the bifurcation shift due to the near-resonant perturbation is in the direction which tends to *induce* the instability. This is exactly what was observed experimentally (Fig. 1).

The exact solution to the cubic [Eq. (3)] is

$$\mu_*=S_++S_-\,,$$

where

$$S_\pm=\left[-3\varepsilon^2\pm\left(\frac{64}{27}\delta^6+9\varepsilon^4\right)^{1/2}\right]^{1/3}. \tag{4}$$

In the small detuning limit ($\delta\rightarrow 0$) Eq. (4) provides a simple scaling law relating the shifted bifurcation point to the perturbation signal,

$$\mu_*=(-6)^{1/3}(\varepsilon)^{2/3}. \tag{5}$$

To test Eqs. (4) and (5) experimentally, we need to relate the normal form parameters μ, ε, δ with the corresponding experimental parameters h_0, h_1, and Δ. In principle, a center manifold reduction would determine the need proportionality constants. In our case, since we do not even have the original dynamical equations, deriving the constants is impossible. In fact, only two free constants are needed: In terms of the experimental parame-

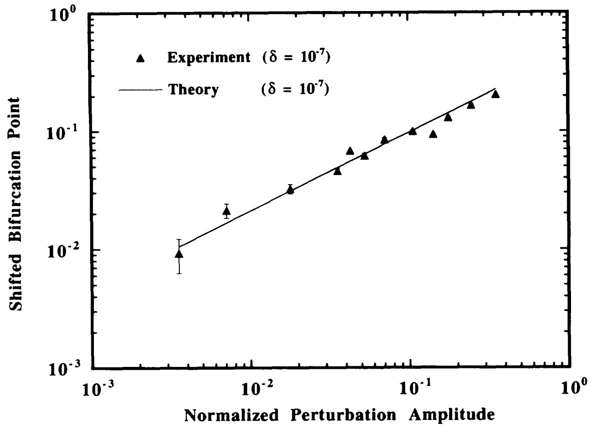

FIG. 2. Scaling law relating the shifted bifurcation point to the strength of the perturbation signal is shown for extremely small detuning frequency, $\delta=10^{-7}$. The data are fitted by 2/3 power law predicted by Eq. (5). The shifted bifurcation point μ and the normalized perturbation amplitude ε are related to the experimental parameters, pump amplitude h_0, and perturbation signal amplitude h_1 as follows: $\mu=[(h_0-h_{0c})/h_{0c}]K_1'$ and $\varepsilon=(h_1/h_0)K_2'$ where h_{0c} is the bifurcation point of the unperturbed system.

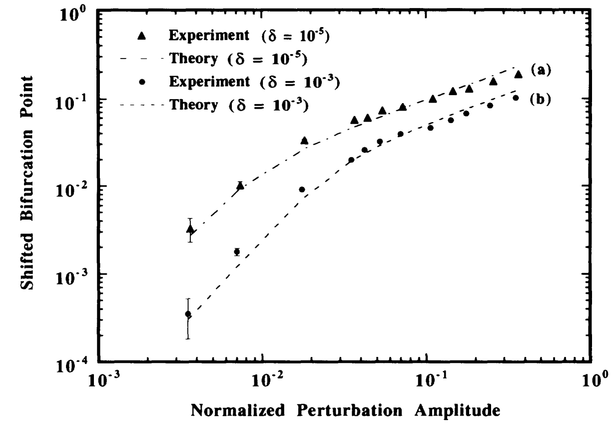

FIG. 3. Experimental results showing the shifted bifurcation point to the perturbation signal strength and the fit by Eq. (4) for somewhat larger detunings (a) $\delta=10^{-5}$ and (b) $\delta=10^{-3}$.

ters the most general form would be

$$\frac{dy}{dt}=\left\{\frac{h_0-h_{0c}}{h_{0c}}\right\}k_1y+k_4y^3+\frac{h_1k_3}{h_{0c}}\cos\left\{\frac{\omega_1-\omega_0/2}{\omega_0}k_2t\right\},$$

where h_{0c} is the bifurcation point of the unperturbed system and k_1, k_2, k_3, and k_4 are arbitrary constants relating μ, δ, and ε to h_0, Δ, and h_1. By proper change of variables this equation can be rewritten as

$$\dot{x}=\mu x+x^3+\varepsilon\cos(\delta t),$$

where $\mu=[(h_0-h_{0c})/h_{0c}]k_1/k_2=[(h_0-h_{0c})/h_{0c}]K_1'$ and $\varepsilon=(h_1/h_0)(k_3/k_2)(k_4/k_2)^{1/2}=(h_1/h_0)K_2'$. The constants K_1' and K_2' can be determined from one set of data relating perturbation amplitude with the bifurcation shift and then be kept "fixed" for all subsequent data sets (i.e., detunings). The results depicting the fit to the theory are shown in Figs. 2 and 3. In Fig. 2 the shifted bifurcation point as a function of perturbation amplitude is shown for extremely small detuning, $\Delta/f_0=\delta=10^{-7}$. The experimental data plotted on a log-log scale and fitted by Eq. (5) fall on a straight line with a slope equaling 2/3.

As the detuning increases the "pure" scaling law [Eq. (5)] is replaced by a functional dependence, $\mu(\varepsilon,\delta)$ described by Eq. (4). Figure 3 shows experimental results for the bifurcation shift as a function of perturbation amplitude for larger detunings, $\Delta/f_0=\delta=10^{-5}$ and $\Delta/f_0=\delta=10^{-3}$. The experimental data shown in Fig. 3 were fitted by Eq. (4). The theory does an excellent job of following the data. The pure power law scaling behavior is replaced by two regimes: (i) extremely small perturbation where the bifurcation shifts as a power of 2 with respect to h_1 and (ii) large perturbations where the 2/3 power law is retrieved. These two regimes were recovered by taking the limit $\varepsilon\rightarrow$ small and $\varepsilon\rightarrow$ large with $\delta\neq0$ in Eq. (4). The range of perturbation signal strength in Figs. 2 and 3 spans over 2 orders of magnitude while the detuning spans a range of 4 orders of magnitude.

One of the applications of these ideas can be in the area of bistable devices. The approach can be utilized to "tune" the switching characteristics of optical or electrical bistable systems, for instance, to reduce the switching voltage or to reduce the switching optical power.

In conclusion, we demonstrate for the first time that near-resonant perturbations tend to induce subcritical bifurcation which is in contrast to the effect observed in supercritical bifurcation. The destabilizing shift has been found to be a function of the perturbation signal strength and the detuning frequency. An augmented normal form used to analyze experimental results reveals a "generic" closed form expression describing the shifted bifurcation point to the detuning and the perturbation signal strength. For extremely small detuning a scaling rule elating the bifurcation shift to the perturbation signal strength is retrieved with a critical exponent equal to 2/3. The experimental data obtained for various detuning are fitted by the expressions derived from normal form analysis. The theory agrees quite well with the experimental data over a wide parameter range (several orders of magnitude).

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frequency ω_1 very near to $\omega_0/2$.
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