Tunneling of a Quantized Vortex: Roles of Pinning and Dissipation

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We have performed a theoretical study of the effects of pinning potential and dissipation on vortex tunneling in superconductors. Analytical results are obtained in various limits relevant to experiment. In general we have found that pinning and dissipation tend to suppress the effect of the vortex velocity dependent part of the Magnus force on vortex tunneling.

To address the question of the resistance of a superconductor at zero temperature, a clear understanding of vortex tunneling is needed. Despite ample experimental evidence for vortex tunneling in superconductors [1], the theoretical study is of dimensional analysis in nature [2,3] and controversial [4]. Furthermore, in vortex tunneling there is no comprehensive study of the role played by the Magnus force, a force with a part proportional to the magnitude of vortex velocity but perpendicular to the direction of the velocity. This is a very unsatisfactory situation, in view of the facts that the Magnus force is a general property of a vortex [5] and tunneling is extremely sensitive to dynamics. In this paper we present an

analytical study on vortex tunneling in the presence of the Magnus force, with special attention paid to the effects of pinning and dissipation. A general picture of those effects on vortex tunneling can be obtained from our study.

We start with the Hamiltonian for a vortex. To make the presentation simple, we consider vortex tunneling in a semi-infinite superconductor film which defines the x-y plane. The supercurrent is along the x direction, and the edge of the film is lying on the x axis. The length scale in the present study is always much larger than the size of a vortex core. A vortex can then be regarded as a point particle, and as discussed in Ref. [6] its effective Hamiltonian can be generally written as

$$H = \frac{1}{2M}[\mathbf{P} - q_v\mathbf{A}(\mathbf{r})]^2 + V(\mathbf{r}) + \sum_j \left[\frac{1}{2m_j}\mathbf{P}_j^2 + \frac{1}{2}m_j\omega_j^2 \left(\mathbf{q}_j - \frac{c_j}{m_j\omega_j^2}\mathbf{r} \right)^2 \right] , \tag{1}$$

with the vector potential $\mathbf{A}$ determined by $\nabla \times \mathbf{A} = h\rho_s d\hat{z}/2$. The meaning of each term in Eq. (1) is as follows. The vector potential reflects the existence of the vortex velocity dependent part (VVDP) of the Magnus force $\mathbf{F}_m = q_v h\rho_s d(\mathbf{v}_s - \dot{\mathbf{r}}) \times \hat{z}/2$. The Magnus force depends on the relative velocity between the superfluid velocity $\mathbf{v}_s$ and the vortex velocity $\dot{\mathbf{r}}$, and the superfluid velocity dependent part (SFVDP) of the Magnus force will contribute to the vortex potential V. In accordance with the calculation of tunneling, the vector potential will be taken as

$$A = \frac{h}{2}\rho_s d(y,0,0) , \tag{2}$$

and it can be shown that results are independent of the choice of gauge. Here $q_v = +1$ (-1) stands for the vorticity parallel (antiparallel) to the unit vector $\hat{z}$ in the z direction, h is the Planck constant, ρ_s is the superfluid electron number density with the factor $\frac{1}{2}$ counting for the Cooper pairing, and d is the thickness of the film. We take the vortex mass M to be finite as demonstrated in Ref. [6], and will show that it is relevant to vortex tunneling. The vortex potential V contains both the contributions from the SFVDP Magnus force and pinning centers. In the following we shall take V to be of the form

$$V(\mathbf{r}) = V_1(y) + \frac{1}{2}k_x x^2 , \tag{3}$$

which allows an extensive analytical study. The pinning potential of a pinning center in the x direction is approximated by the harmonic potential, and k_x should be determined experimentally. The potential $V_1(y)$ consists of the contributions from the SFVDP Magnus force and the pinning potential in the y direction, which has a metastable point at $y=0$ in the present paper. In the limit of no pinning (an ideal semi-infinite film) and extreme type II,

$$V_1(y) = \frac{h}{2}\rho_s d\left(-v_0 y + \frac{h}{2m_c}\ln(y) \right) , \tag{4}$$

only the contribution from the SFVDP Magnus: the potential due to external supercurrent v_0 and the image potential from the edge of the superconductor film. Here m_c is the mass of a Cooper pair. The dissipative environment of the vortex, the last term in Eq. (1), consists of a set of harmonic oscillators as formulated in Ref. [7]. The effect of the dissipative environment is specified by the spectral function

$$J(\omega) \equiv \pi \sum_j \frac{c_j^2}{2m_j\omega_j}\delta(\omega - \omega_j) . \tag{5}$$

In the present paper, we shall assume the spectral function to have the following form:

$$J(\omega) = \eta \omega^s \exp\{-\omega/\omega_c\}, \tag{6}$$

with ω_c the cutoff frequency whenever needed. In accordance with Ref. [7], $s > 1$ is the super-Ohmic case, $s = 1$ the Ohmic case, and $0 \leq s < 1$ the sub-Ohmic case. In the Ohmic damping case, η is the vortex friction such as discussed in Ref. [8]. Although the problem here is formulated for a vortex tunneling from an edge into a film, a problem directly related to the decay of a supercurrent at zero temperature, the present formulation is equally applicable to the case of a vortex tunneling out of any metastable well with a potential approximated by Eq. (3). In

that case, the form of $V_1(y)$ may be completely different from Eq. (4).

A remark on the relationship between the present study and others is in order. We note that the vortex motion is identical to the motion of an electron in the presence of a magnetic field [9]. Results obtained in the present paper can then be directly applied to that case with q_v replaced by electron charge e and $h\rho_s/2$ by magnetic field B [10]. For the case of a vortex in superfluid ^{4}He, the superfluid electron pair number density $\rho_s/2$ should be replaced by the superfluid helium atom number density ρ_s [11].

The tunneling is described by the Euclidean action [7,9,10]

$$S = \int_0^{\hbar\beta} d\tau \left\{ \frac{1}{2}M\dot{\mathbf{r}}^2 + i\frac{h}{2}q_v\rho_s d\dot{x}y + V_1(y) + \frac{1}{2}k_x x^2 + \sum_j \left[\frac{1}{2}m_j\dot{q}_j^2 + \frac{1}{2}m_j\omega_j^2 \left(q_j - \frac{c_j}{m_j\omega_j^2}\mathbf{r} \right)^2 \right] \right\}, \tag{7}$$

where $\beta = 1/k_B T$ is the inverse temperature. The tunneling rate is equal to $\exp\{-S_c/\hbar\}$, where the semiclassical action S_c is determined by the bounce solution of the equation $\delta S = 0$, in which the periodic boundary condition, $(\mathbf{r}(\hbar\beta), \{q_j(\hbar\beta)\}) = (\mathbf{r}(0), \{q_j(0)\})$, is required.

We are interested in the vortex tunneling out of the metastable state $y = 0$. After the tunneling in the y direction other degrees of freedom can take arbitrary value. Therefore the summation over the final states, integrations over environmental degrees of freedoms, $\{q_j\}$, and over the x coordinate, will be taken. The effective Euclidean action after the integrations is

$$S_{\text{eff}} = \int_0^{\hbar\beta} d\tau \left[\frac{1}{2}M\dot{y}^2 + V_1(y) \right] + \frac{1}{2}\int_0^{\hbar\beta} d\tau \int_0^{\hbar\beta} d\tau' [k(|\tau - \tau'|) + g(\tau - \tau')][y(\tau) - y(\tau')]^2, \tag{8}$$

with the normal damping kernel $k(\tau)$ as

$$k(\tau) = \frac{1}{\pi}\int_0^\infty d\omega J(\omega) \frac{\cosh[\omega(\hbar\beta/2 - \tau)]}{\sinh[\omega\hbar\beta/2]}, \tag{9}$$

and the damping kernel $g(\tau)$ due to the VVDP Magnus force coupled x-direction motion, which we shall call the anomalous damping kernel, as

$$g(\tau) = \frac{1}{\hbar\beta}\frac{1}{2M}\left(\frac{h}{2}\rho_s d \right)^2 \sum_{n=-\infty}^{\infty} \frac{M\omega_x^2 + \xi_n}{Mv_n^2 + M\omega_x^2 + \xi_n}e^{iv_n\tau}. \tag{10}$$

Here

$$\xi_n = \frac{1}{\pi}\int_0^\infty d\omega \frac{J(\omega)}{\omega}\frac{2v_n^2}{\omega^2 + v_n^2}, \tag{11}$$

$v_n = 2\pi n/\hbar\beta$, and $\omega_x^2 = k_x/M$. In the large τ limit, from Eqs. (6) and (9) the normal damping kernel takes the form

$$k(\tau) = \frac{1}{\pi}\eta\frac{1}{\tau^{s+1}}, \tag{12}$$

which demonstrates the one-to-one correspondence between the low frequency part of the spectral function and the long time behavior of the damping kernel. Now we have obtained an effective one-dimensional problem, Eq. (8), with the normal and anomalous damping kernels $k(\tau)$ and $g(\tau)$. This is the starting point for the following detailed study of the tunneling in various experimentally accessible limits.

No pinning, no dissipation.— Without disorder and dissipation, there is no quantum decay of supercurrent in an infinite film [12]. In reality such as discussed in the present paper the film always has an edge; tunneling is possible. To see the role played by the VVDP Magnus force, we first study the case with no dissipation and no pinning. Because of the VVDP Magnus force, motions in the x and y directions become coupled to each other. Therefore although here the normal damping kernel $k(\tau) = 0$, the $n = 0$ mode in the anomalous kernel $g(\tau)$ contributes to the effective action. Then the effective action is

$$S_{\text{eff}} = \int_0^{\hbar\beta} d\tau \left[\frac{1}{2}M\dot{y}^2 + V_1(y) \right] + \frac{1}{4M}\left(\frac{h}{2}\rho_s d \right)^2 \frac{1}{\hbar\beta}\int_0^{\hbar\beta} d\tau \int_0^{\hbar\beta} d\tau' [y(\tau) - y(\tau')]^2. \tag{13}$$

We note that in the language of spectral function, this corresponds to the sub-Ohmic bath case with $s = 0$ [cf. Eqs. (12) and (8)], and Eq. (13) is explicitly gauge invariant under the change $y \to y + \text{const}$, because of the periodic boundary condition of x imposed in the tunneling calculation.

For low enough density ρ_s, that is, for a weak VVDP Magnus force, there is a tunneling solution. Particularly, for very low density and low temperatures the semiclassical action may be evaluated perturbatively:

$$S_c = \int_{-\infty}^\infty d\tau \left[\frac{1}{2}M\dot{y}_c^2(\tau) + V_1(y_c(\tau)) \right] + \frac{1}{2M}\left(\frac{h}{2}\rho_s d \right)^2 \left[\int_{-\infty}^\infty d\tau\, y_c^2(\tau) - \frac{1}{\hbar\beta}\left(\int_{-\infty}^\infty d\tau\, y_c(\tau) \right)^2 \right], \tag{14}$$

where $y_c(\tau)$ is the bounce solution at zero temperature without the VVDP Magnus force. Equation (14) shows a remarkable ρ_s^2 dependence and linear temperature dependence [13].

For high enough density, that is, for a strong VVDP Magnus force, the tunneling rate vanishes at zero temperature, because the VVDP Magnus force renormalizes the original potential such that the state near $y=0$ is stable. This can be noted from Eq. (13) where as $T \to 0$ the cross term in the last term disappears. A straightforward calculation leads to the criterion for the localization as

$$\frac{1}{M}\left[\frac{h}{2}\rho_s d\right]^2 > \left|\frac{d^2 V_1(y)}{dy^2}\right|_{\text{barrier top}}. \tag{15}$$

Thus we have obtained that the $s=0$ dissipative environment is marginal for localization in tunneling decay, compared to the $s=1$ for the case of tunneling splitting [14]. This is the dynamical localization caused by the VVDP Magnus force [12]. It is worthwhile to point out that according to Eq. (15) although a large VVDP Magnus force inhibits vortex tunneling, a large vortex mass instead favors vortex tunneling.

No pinning, finite dissipation.— Now we set the pinning potential $k_x/M = \omega_x^2 = 0$ in Eq. (10). But the normal damping kernel $k(\tau)$ and the anomalous damping kernel $g(\tau)$ are finite. Since the quantum tunneling corresponds to the limit of large imaginary time, $\hbar\beta \to \infty$,

we look for the large time limit behavior of the anomalous damping kernel $g(\tau)$. In this limit, we may replace the summation $1/\hbar\beta\Sigma_n$ by the integration $(1/2\pi)\int dv$. Then for the environment $0 < s < 2$, we find the anomalous damping kernel $g(\tau)$ in the large τ limit as

$$g(\tau) = a\frac{1}{2\pi M}\left[\frac{h}{2}\rho_s d\right]^2 \frac{1}{\bar\eta}\frac{1}{\tau^{2-s+1}}, \tag{16}$$

with a a numerical constant of order unity. Here

$$\bar\eta = \frac{2}{\pi}\frac{\eta}{M}\int_0^\infty dz\frac{z^{s-1}}{z^2+1}, \tag{17}$$

and $J(\omega) = \eta\omega^s$ has been used. The effective dissipative environment corresponding to the anomalous damping kernel is $s_{\text{eff}} = 2-s$ [cf. Eqs. (6) and (12)], which leaves the Ohmic damping unchanged, transforms the sub-Ohmic damping into super-Ohmic damping, and vice versa.

For the case $s > 2$, using Eq. (6) for the spectral function $J(\omega)$, we find that the effective dissipative environment corresponding to the anomalous damping kernel is $s_{\text{eff}} = 0$, which smoothly connects the result for $0 < s < 2$.

An important example is the Ohmic damping case, where an exact expression for a in Eq. (16) can be obtained. Carrying out a detailed but straightforward calculation, we find the effective action at zero temperature as

$$S_{\text{eff}} = \int_{-\infty}^\infty d\tau\left[\tfrac{1}{2} m\dot{y}^2 + V_1(y)\right] + \frac{1}{2\pi}\eta_{\text{eff}}\int_{-\infty}^\infty d\tau\int_{-\infty}^\infty d\tau'\frac{1}{|\tau - \tau'|^2}[y(\tau) - y(\tau)]^2, \tag{18}$$

with the effective damping strength η_{eff} as

$$\eta_{\text{eff}} = \eta + \left[\frac{h\rho_s d}{2}\right]^2\frac{1}{\eta}. \tag{19}$$

The effect of a dissipative environment can now be summarized as follows. It has been demonstrated in Ref. [14] that the sub-Ohmic dissipation has strong effects on tunneling, while the super-Ohmic dissipation has weak effects. From the above analysis we have that if the normal damping kernel $k(\tau)$ is sub-Ohmic, the anomalous damping kernel $g(\tau)$ is super-Ohmic, and vice versa. Then according to Ref. [14] the effect of the VVDP Magnus force, represented by the anomalous damping kernel, is weak (strong) on vortex tunneling for the normal sub-Ohmic (super-Ohmic) damping. In particular,

for normal super-Ohmic damping with $s > 2$, the vortex tunnels as if there were no effect of dissipation. For the normal Ohmic damping with $s = 1$, we need to compare the relative strength of the VVDP Magnus force and the dissipation according to Eq. (19). Classically, the VVDP Magnus force tends to keep a vortex moving along an equal potential contour, but the friction instead down the potential to compete with the VVDP Magnus force. In general, we can then conclude that the dissipation tends to suppress the effect of the VVDP Magnus force on vortex tunneling.

Finite pinning, no dissipation.— The normal damping kernel vanishes in this situation. The anomalous damping kernel can be expressed by hyperbolic functions. We find that the effective action is then

$$S_{\text{eff}} = \int_0^{\hbar\beta} d\tau\left[\tfrac{1}{2} M\dot{y}^2 + V_1(y)\right] + \frac{1}{4M}\left[\frac{h}{2}\rho_s d\right]^2\int_0^{\hbar\beta} d\tau\int_0^{\hbar\beta} d\tau'\frac{\omega_x}{2}\frac{\cosh[\omega_x(\hbar\beta/2 - |\tau - \tau'|)]}{\sinh[\omega_x\hbar\beta/2]}[y(\tau) - y(\tau')]^2. \tag{20}$$

It is the super-Ohmic case with $s_{\text{eff}} = \infty$, because the effective spectral function has the form of $\delta(\omega - \omega_x)$, which has no low frequency mode. Therefore the tunneling rate is nonzero for any magnitude of the VVDP Magnus force. This result shows that pinning has a very strong influence on the vortex tunneling in the presence of

the VVDP Magnus force, because the introduction of the pinning potential of a pinning center bends the straight line trajectory of a vortex and makes the transition to other trajectories possible. We note that by letting $\omega_x = 0$ we recover Eq. (13), where the tunneling rate vanishes for a sufficiently strong VVDP Magnus force.

We can evaluate the semiclassical action perturbatively if the density is low or if the pinning potential is strong, as is done in Eq. (14). In the large density limit the semiclassical action may be evaluated by a variational method similar to Ref. [7] in the case of Ohmic damping. However, we have a more powerful method to perform the calculation, because the strong VVDP Magnus force freezes the kinetic energy of the vortex. In this case x and y coordinates now form a pair of canonically conjugate variables, and the Euclidean action is [9,10]

$$S_{\text{eff}} = \int_0^{\hbar\beta} d\tau \left[i\frac{\hbar}{2} q_v \rho_s d\, \dot{x}y + V_1(y) + \frac{1}{2} k_x x^2 \right]. \quad (21)$$

Following the calculation outlined in Refs. [9,10], we find the semiclassical action in the form

$$S_c = \hbar\rho_s d \int_0^{y_t} dy \left[\frac{2V_1(y)}{k_x} \right]^{1/2}, \quad (22)$$

with y_t the turning point determined by the equation $V_1(y) = 0$ [15]. The result shows that as k_x increases, the semiclassical action decreases. Therefore a large pinning potential in the x direction helps the tunneling in the y direction [16].

General case: finite pinning and dissipation.—In general we need to go back to Eqs. (8)–(11) to study the tunneling under the influence of pinning and dissipation. However, based on the insight gained by the above analysis we can draw a general conclusion without a detailed study: Since both dissipation and pinning tend to suppress the effect of the VVDP Magnus force, the total effect of them will do the same. Indeed the anomalous damping kernel is always super-Ohmic. In particular, in the presence of strong pinning and Ohmic damping, the super-Ohmic-like anomalous damping kernel $g(\tau)$ may be ignored compared to the Ohmic normal damping kernel $k(\tau)$. Then from Eq. (8) we have the effective action as

$$S_{\text{eff}} = \int_0^{\hbar\beta} d\tau [\tfrac{1}{2} M\dot{y}^2 + V_1(y)] + \frac{\eta}{2\pi} \int_0^{\hbar\beta} d\tau \int_0^{\hbar\beta} d\tau' \frac{1}{|\tau - \tau'|^2} [y(\tau) - y(\tau')]^2, \quad (23)$$

which looks as if there were no effect of the VVDP Magnus force. This may explain the pronounced experimental observation of the absence of the effect of the VVDP Magnus force in vortex tunneling experiments on dirty superconductors [1]. It is worthwhile to point out that, stimulated by Ref. [7], Eq. (23) has been studied extensively.

To summarize, we have performed a complete analytical study of the influence of pinning and dissipation on vortex tunneling. The VVDP Magnus force tends to decrease the tunneling rate. In the absence of pinning and dissipation, there is no tunneling for a strong VVDP Magnus force. Detailed results have been obtained when one or all of them are absent. Both pinning and dissipation tend to suppress the influence of the VVDP Magnus force on vortex tunneling. Present results may explain the absence of effects of the VVDP Magnus force in vortex tunneling experiments on dirty superconductors.

[1] A. V. Mitin, Zh. Eksp. Teor. Fiz. **93**, 590 (1987) [Sov. Phys. JETP **66**, 335 (1987)]; A. C. Mota, P. Visani, and A. Pollini, Phys. Rev. B **37**, 9830 (1988); N. Giordano, Phys. Rev. Lett. **61**, 2137 (1988); A. C. Mota, G. Juri, P. Visani, A. Pollini, T. Teruzzi, K. Aupke, and B. Hilti, Physica (Amsterdam) **185-189C**, 343 (1991); L. Fruchter, A. P. Malozemoff, I. A. Campbell, J. Sanchez, M. Konczykowski, R. Griessen, and F. Holtzberg, Phys. Rev. B **43**, 8709 (1991); Y. Liu, D. B. Haviland, L. Glazman, and A. M. Goldman, Phys. Rev. Lett. **68**, 2224 (1992); D.

Prost, L. Fruchter, I. A. Campbell, N. Motohira, and M. Konczykowski, Phys. Rev. B **47**, 3457 (1993); J. Tejada, E. M. Chudnovsky, and A. Garcia, ibid. **47**, 11552 (1993).

[2] G. Blatter, V. B. Geshkenbein, and V. M. Vinokur, Phys. Rev. Lett. **66**, 3297 (1991); G. Blatter and V. B. Geshkenbein, Phys. Rev. B **47**, 2725 (1993).

[3] M. P. A. Fisher, T. A. Tokuyasu, and A. P. Young, Phys. Rev. Lett. **66**, 2931 (1991).

[4] P. Ao, Phys. Rev. Lett. **69**, 2997 (1992).

[5] P. Ao and D. J. Thouless, Phys. Rev. Lett. **70**, 2158 (1993); P. Ao, Q. Niu, and D. J. Thouless, Physica B&C (to be published).

[6] Q. Niu, P. Ao, and D. J. Thouless (to be published).

[7] A. O. Caldeira and A. J. Leggett, Ann. Phys. (N.Y.) **149**, 374 (1983); **153**, 445(E) (1984).

[8] J. Bardeen and M. J. Stephen, Phys. Rev. **140**, A1197 (1965).

[9] D. J. Thouless, P. Ao, and Q. Niu, Physica A (to be published).

[10] J. K. Jain and S. Kivelson, Phys. Rev. A **36**, 3467 (1987); Phys. Rev. B **37**, 4111 (1988); H. A. Fertig and B. I. Halperin, ibid. **36**, 7969 (1987).

[11] C. M. Muirhead, W. F. Vinen, and R. J. Donnelly, Philos. Trans. R. Soc. London A **311**, 433 (1984).

[12] P. Ao, J. Low Temp Phys. **89**, 543 (1992).

[13] The same result was obtained for the electron tunneling in the presence of a magnetic field by P. Ao, Mod. Phys. Lett. B **7**, 927 (1993).

[14] A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, A. Garg, and W. Zwerger, Rev. Mod. Phys. **59**, 1 (1987).

[15] A special case of Eq. (22) was obtained by M. V. Feigel'man, V. B. Geshkenbein, A. I. Larkin, and S. Levit, JETP Lett. **57**, 711 (1993).

[16] This result was obtained in Ref. [9].