Integrable Nonlinear Accelerator Lattices

Carson C. Chow and John R. Cary
Department of Astrophysical, Planetary, and Atmospheric Sciences, Boulder, Colorado 80309-0391
(Received 4 June 1993)

A method for creating integrable nonlinear accelerator lattices is presented. Fixed points for the two-dimensional return map corresponding to the lattice are found. Minimizing the residues (an indicator of the size of the associated island) of these fixed points by varying lattice parameters eliminates large islands and regions of chaos. The resulting nonlinear systems have larger dynamical apertures and are more stable to perturbations induced by, for example, error fields.

PACS numbers: 41.85.-p, 05.45.+b, 29.20.-c

We present a nonperturbative method for designing nonlinear, integrable accelerator lattices. Our lattices have the nonlinearities required for eliminating chromaticity, but these nonlinearities are structured so that islands and chaotic regions, which cause particle loss, are not present. The nonlinearity of our lattices will make them more stable to perturbations due to error fields. By removing the islands we obtain lattices with larger dynamic apertures. To illustrate our method, we show that adding octupoles and decupoles of particular strengths and at particular locations to the lattice of the Advanced Light Source at Lawrence Berkeley Laboratory eliminates islands and increases the dynamic aperture.

Linear lattices have been considered desirable, in part because linear dynamics is easily understood. In a linear lattice, all trajectories execute linear oscillations about the design orbit, the closed orbit at the center of the beam, at a frequency known as the *tune*, when normalized by the frequency of motion around the ring. This linear motion conserves the Courant-Snyder invariant [1]. This implies that the intersections of the trajectory with a transverse plane in phase space lie on ellipses.

However, linear motion is unstable to perturbations. With a resonant perturbation, i.e., one whose periodicity matches that of the orbit, particles at some phase receive an outward kick on each circuit. Because the tune is independent of amplitude, the outwardly kicked particle remains in resonance until it leaves the machine. To some extent the situation is saved by the Kolmogorov-Arnold-Moser (KAM) theorem [2], which guarantees a region of confined trajectories of finite measure near the design orbit, provided the central tune is not a multiple of 1/3 or 1/4. For this reason accelerators are designed with lattices having central tunes far from low-order rationals.

To keep the tune far from the low-order rationals for the range of energies in the beam, accelerators are designed to have vanishing chromaticity, the variation of the tune with particle energy, by adding sextupole magnets. However, sextupoles are nonlinear elements. As a result, the trajectories launched successively farther from the design orbit are seen to comprise island structures or not to be confined at all. This, consequently, limits the dynamic aperture, the confined region of phase space.

Thus, designing towards linearity is self-contradictory. Linearity with perturbations requires the tune to be far from low-order rationals. Keeping the tune far from low-order rationals over a range of energies by adding sextupoles ruins linearity. To escape this logic, we propose that accelerator lattices be made nonlinear, yet integrable, so that good confinement remains. In this Letter we show how this can be done.

In our method, the nonlinearity needed to eliminate chromaticity is retained, so that transverse phase space structure does not vary with energy, but we modify the nonlinearity so that it does not cause the invariant surfaces to break up into islands and chaos. Our method does not rely on perturbation theory, and so works for arbitrarily large dynamic aperture. The result of our method is an integrable, nonlinear transverse focusing system, which, by nature of being nonlinear, is stable to perturbations [2] and has larger dynamic aperture than the unoptimized system.

Our method is so far limited to systems of 1.5 degrees of freedom. Thus, we concentrate on the transverse motion parallel to the bending plane of the ring and, consequently, ignore the energy oscillations and the coupling to the vertical motion. In this case the motion reduces to a Hamiltonian of 1.5 degrees of freedom with coordinates (x, p, s), where s is the arc length along the ring.

The dynamics of this system is embodied in the return map. The return map,

$$z' = T(z), \tag{1}$$

is obtained by following a trajectory from a given transverse plane, at which the transverse coordinates are $z \equiv (x, p)$, to its location z' in the same plane after one circuit. Since the system is Hamiltonian, the mapping T is symplectic [3]. For an integrable lattice, the successive iterates of the return map for any initial condition lie on what is topologically a circle and have a rate of rotation about the center given by the tune, which is dependent on the amplitude of oscillation if the dynamics is nonlinear.

For the typical nonintegrable lattice, the iterates near the design orbit lie on the Courant-Snyder invariant curves and rotate about the design orbit at a rate given by the *central tune*. The KAM theorem states that there exist invariant circles with tune irrational and different from the central tune at finite distance from the design orbit. (This difference is known as the *nonlinear tune shift*.) For rational tunes island chains replace invariant circles. Associated with these island chains are p/q fixed points, which map onto themselves after q iterations, after which they have encircled the design orbit p times. We say that such a fixed point has *rotational tune* p/q (as opposed to the tune related to the rate of rotation about this fixed point, known as the *island tune*). Beyond some distance there are no more invariant circles; regular structures in phase space disappear, chaos appears, and the orbits launched in this region are not confined.

Greene's work [4] shows that the appearance of chaos is related to the linear behavior of orbits of nearby fixed points. This linear behavior is found by substituting $z \to z^0 + \delta z$ into the mapping Eq. (1), where z^0 is the fixed point, and keeping only up to first order in δz. Thus,

$$\delta z^q = M(z^{q-1})M(z^{q-2})\cdots M(z^0)\delta z \tag{2}$$

$$\equiv M^q(z^q)\delta z, \tag{3}$$

where

$$z^l = T(z^{l-1}), \tag{4}$$

and the evolution of nearby trajectories is given by the *tangent map*,

$$M(z^l) \equiv \left.\frac{\partial T}{\partial z}\right|_{z=z^l}. \tag{5}$$

The product M^q is a linear 2×2 matrix with determinant equal to 1 (since the map is area preserving). The eigenvalues are given by

$$\lambda_\pm = \frac{\mathrm{Tr}(M^q) \pm \sqrt{[\mathrm{Tr}(M^q)]^2 - 4}}{2}, \tag{6}$$

where $\mathrm{Tr}(M)$ is the trace of matrix M. The eigenvalues are either both on the unit circle and complex conjugate (elliptic point), or they are real and reciprocal (hyperbolic point). A convenient quantity is the residue R,

$$R \equiv \frac{2 - \mathrm{Tr}(M^q)}{4}. \tag{7}$$

If $0 < R < 1$ the fixed point is elliptic, if $R < 0$ the point is hyperbolic, and if $R > 1$ the point is hyperbolic with reflection.

Greene's postulate [4] is that the existence of an invariant circle of irrational tune ν can be determined by examining the residues of the sequence of fixed points $(z_1, z_2, \ldots)$ with rotational tunes $(p_1/q_1, p_2/q_2, \ldots)$ that are the successive continued-fraction [5] approximants to ν. In particular, an invariant circle with tune

$$\nu = \lim_{n\to\infty} \nu_n \tag{8}$$

exists if

$$\lim_{n\to\infty} R_n = 0. \tag{9}$$

If the invariant surface does not exist, the residue limit will diverge to infinity. Thus, local behavior of fixed points near a surface of irrational tune determine the existence of an invariant surface having that irrational tune. In addition, renormalization theory [6] shows that the low-order fixed points control the behavior of the higher-order ones. Hence, zeroing the residues of a few fixed points can be enough to guarantee an invariant circle near that tune.

This is the basis for our optimization method. We add nonlinear elements that allow us to reduce the residues of the main fixed points. (A similar type of optimization has been used successfully for making integrable the flow of magnetic field lines of stellarators [7].) Our reduction of the residues is carried out with the constraint that the chromaticity of the design orbit remain zero. This helps prevent phase space from varying significantly with particle energy. (In certain lattices, changes in longitudinal momentum can create island chains near the origin which are disastrous for particle confinement [8].) We also reduce the fixed-point chromaticities for the same reason.

The procedure begins by examining the surface of section to determine the critical fixed points. A set of *residuals*, $\mathbf{F}$, to be zeroed are defined. Examples of residuals are the difference between the residue of the design orbit and the desired value, the residue of an outer fixed point, the chromaticity of the design orbit, and the difference between location of an outer fixed point and its desired location. A few nonlinear elements (sextupoles, octupoles, decupoles) are inserted at arbitrary locations in the lattice. We then vary a subset $\mathbf{v}$ of the parameters of the accelerator in order to zero these residuals, $\mathbf{F}(\mathbf{v})$, which are functions of the parameter set. Thus, we wish to solve the system,

$$\mathbf{F}(\mathbf{v}) = 0, \tag{10}$$

of equations. This system of equations is solved iteratively by a multidimensional Newton's method. We solve the system,

$$J \cdot \delta\mathbf{v} = -\mathbf{F}(\mathbf{v}), \tag{11}$$

of linear equations, for the change $\delta\mathbf{v}$ of the parameter set that moves the set closer to satisfying Eq. (10). The matrix

$$J_{ij} \equiv \frac{\partial F_i}{\partial v_j} \tag{12}$$

is the Jacobian matrix. For complicated lattices J can be determined numerically. Often it is convenient to have more unknowns than equations, in which case the matrix equation (12) is solved by singular valued decomposition [9].

To facilitate this method a Motif [10] based X windows

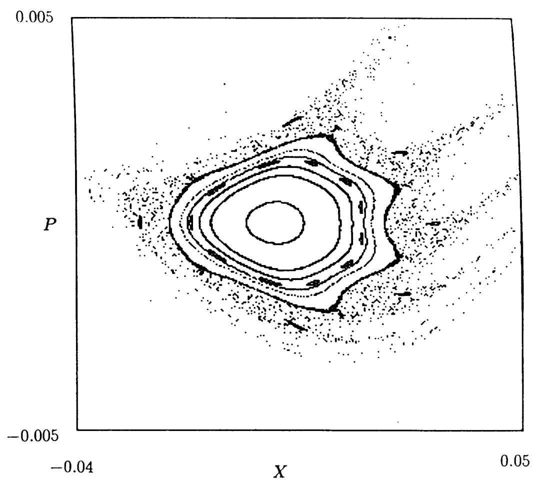

FIG. 1. Surface of section for the ALS.

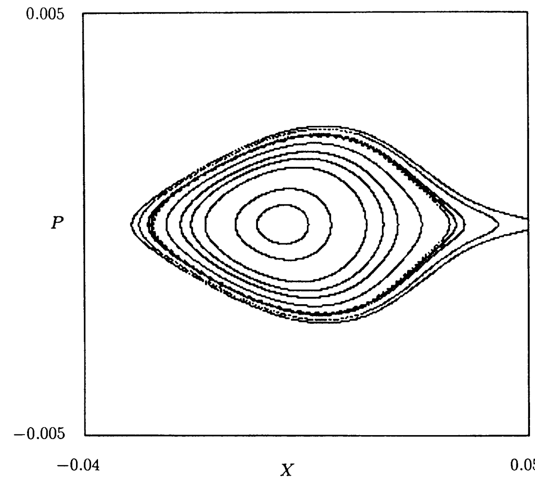

FIG. 2. Optimized surface of section for the ALS.

software tool has been developed. The tool creates the map and the tangent map from the lattice elements and their positions. It can produce a surface of section. Initial conditions are typed in or selected with a mouse pointer. For each orbit the tune and continued fraction expansion of the tune is printed out. Fixed points of arbitrary order are found by Newton's method, with initial guesses selected by the mouse on the surface of section. The user selects residuals based on the fixed points found. The code calculates the optimizing step satisfying Eq. (11), then recomputes the location and attributes of the new fixed points corresponding to the new parameters.

As an example, this technique was applied to the Advanced Light Source (ALS) [11]. The ALS ring has twelve equivalent superperiods, each of which contains three dipole bending magnets, six quadrupoles for focusing, and four sextupoles for chromaticity control. The superperiod has front to back symmetry. Thus, the parameters for the elements of only half of the superperiod are needed.

The superperiod map and tangent map are constructed by composing the half superperiod map with itself reversed. The maps for the linear elements (drifts, bends, and quadrupoles) were obtained by exact integration. The maps for the nonlinear elements were obtained by sandwiching a thin lens between two drifts of length equal to half of that of the magnet. This is equivalent to lowest-order symplectic integration through a thick lens (cf. [12], Eqs. 5.43 and 5.44). Lattice parameters were provided by Forest [8]. We verified our code by showing that it correctly calculates the central tune, $\nu_c = 0.189\,754\,67\ldots$, and chromaticity of the ALS. Figure 1 shows the surface of section for the original lattice. There is a 2/11 island chain, and a 1/6 chain in a chaotic region.

Into half a superperiod of the ALS lattice we inserted four octupoles and two decupoles. This choice was arbitrary, and, depending on the specific situation, different types and numbers of nonlinear elements could be used equally well. We then reduced the residues of the 2/11 island chains keeping the central tune and chromaticity fixed. As the 2/11 island residues decreased, the chain disappeared and the 1/6 chain was drawn into the center. Invariant circles formed outside of the chain, and the region became much less chaotic. The dynamic aperture was increased by moving the 1/6 island chain away from the center by using as a residual the difference between the position of one of the fixed points and its desired position. The residue and the island chromaticity were reduced simultaneously. In our code, island chromaticity (ζ) was defined to be the variation, $\zeta_R = dR/d\delta$, of the residue with respect to the normalized longitudinal momentum δ. This definition avoids the divergence that occurs when the fixed-point tune goes to zero, and it is sensible for both elliptic and hyperbolic fixed points.

The optimized surface of section is shown in Fig. 2. The island chains have been eliminated, and the phase space chromaticity has been zeroed as much as possible, so that the entire phase space, as opposed to just the central tune, is invariant to small changes in momentum. The thin lens kick strengths and locations of the inserted nonlinear elements are given in Table I with length unit of meters and momentum unit such that the design longitudinal momentum p_0 is unity.

To see what these numbers mean physically, we note

TABLE I. Parameters of nonlinear elements.

Type	Position (m)	Kick	Field (T)
Octupole	4.550	-295.04	-0.92
Octupole	5.752	-247.59	-0.77
Decupole	5.902	-376.03	-0.06
Octupole	5.952	1082.80	3.38
Decupole	6.767	-955.27	-0.15
Octupole	7.397	-900.10	-2.81

that the magnetic rigidity p_0c/e of the ALS is 5 Tm. For a thin lens, the kick coefficient is $K = bL/a^m$, where b is the strength of the magnetic field at the pole face, L is the length of the pole face, a is the half aperture, and m is 2 for sextupoles, 3 for octupoles, and 4 for decupoles. For a pole-face half aperture of 0.05 m and a pole face length of 0.2 m the fields are given in Table I. These fields are seen to be of reasonable magnitude.

This work shows that integrable nonlinear lattices can be obtained, at least within the limitation of an analysis that keeps only 1 transverse degree of freedom. As such, this method should apply to electron machines, in which the beams are flat and the vertical oscillations are nearly uncoupled to the horizontal. These methods may also be of use to machines with round beams, as the detrimental chaotic effects of the coupled system should be smaller when the uncoupled system is made integrable. We are currently developing methods to test this hypothesis.

We wish to thank E. Forest for his advice, encouragement, and assistance, and his suggestion that this work may be particularly applicable to electron machines. We also thank J. Hanson for advice and useful discussions. This research was supported by the Department of Energy Grant No. DE-FG02-86ER40302.

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