Controlling Extended Systems of Chaotic Elements

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A control scheme is devised for taming the dynamics of convectively unstable extended systems exhibiting chaotic behavior. We show how complex spatiotemporal phenomena can be eliminated in arrays of coupled chaotic elements, in favor of a coherent state in which all elements are synchronized to a prescribed periodic orbit of the uncoupled system. The propagation of the synchronizing front and the resulting steady state properties of the array are analyzed in the presence of noise.

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Spatially extended nonlinear systems often display complex time evolutions whose description cannot be captured by a low-dimensional dynamical model. In many applications, it is advantageous to avoid phenomena such as spatiotemporal chaos and turbulence, yet to operate the system at external conditions for which these are the natural states. In open flow systems, noise is often of paramount importance in the development of nonlinear waves, through the presence of a convective instability. There, a single local disturbance is amplified as it is convected through the spatial extent until eventually it flows out of the system; no permanent structure is produced. However, a persistent noise can have a devastating effect on the eventual dynamics through the appearance of sustained complex wave patterns downstream [1,2]. These *noise-sustained* structures have been characterized by effectively one-dimensional models in a variety of convectively unstable systems, such as viscous films flowing down an incline [3], Taylor-Couette flow with an imposed axial flow [4], and channel flow in the presence of a periodic array of stationary obstacles [5]. Convective instabilities can also arise in discrete spatial arrays of nonlinear oscillators or neurons. Ideally, one is interested in taming the dynamics to behave coherently, by applying only *small* changes to one or several accessible parameters. In this Letter, it is shown for the first time how a convectively unstable chaotic state can be controlled in the presence of noise to evolve periodically in time, uniformly along its spatial extent. The strategy is based on the application of controllers, dispersed periodically in space, with each one perturbing the value of a local parameter, using measurements made only in its neighborhood. Experimentally, the controller actions can be invisioned as external pressure changes applied at locations along the boundary of a continuous flow, or as a small external driving force applied to a small number of the oscillators distributed in a spatial array.

A prototypical system in which a convective instability may lead to intermittent spatiotemporal dynamics can be constructed by forming a one-dimensional array of coupled elements, each of which is described by an identical, low-dimensional chaotic model [6]. The elements interact with their nearest neighbors asymmetrically, modeling a preferred direction of propagation, or a mean flow. In the control scheme introduced here, we show how any orbit of the typically dense set of unstable periodic orbits underlying the low-dimensional chaotic attractor of a *single, uncoupled* element can be synchronized throughout the *entire* array. In the absence of coupling between the elements, such an orbit can be stabilized for a *single uncoupled* element through a recently introduced control procedure for low-dimensional systems [7]. However, it is unfeasible to apply this procedure to large coupled arrays, where the global dynamics does not settle onto a low-dimensional invariant set. In fact, the eventual dynamics often occurs on a set whose dimension is of the order of the number of interacting elements.

For convenience, consider discrete time systems, which are often modeled as coupled map lattices [8]. In a linear array of one-dimensional maps, the scalar state $x_j(n+1)$ (site j, time $n+1$) is determined according to the evolution equations

$$x_j(n+1)=(1-\gamma_1-\gamma_2)f(x_j(n)) + \gamma_1 f(x_{j-1}(n))+\gamma_2 f(x_{j+1}(n)) \qquad (1)$$

for $j=2,\ldots,N-1$, where the nearest neighbor coupling constants obey $\gamma_1>\gamma_2\geq 0$. The nonlinear mapping is taken to be the quadratic map $f(x)=1-ax^2$, with the parameter a chosen well within the chaotic regime. The type of boundary conditions (periodic or open) strongly influences the eventual motion that will be observed [1,9]. For open boundary conditions, say

$$x_1(n+1)=(1-\gamma_2)f(x_1(n))+\gamma_2 f(x_2(n))$$

and

$$x_N(n+1)=(1-\gamma_1)f(x_N(n))+\gamma_1 f(x_{N-1}(n)),$$

a range of the couplings γ_1 and γ_2 exists, for which the system exhibits a stable coherent chaotic state (this is the subject of Ref. [9]). Arbitrarily weak noise destroys the coherent state, due to the fact that it is convectively unstable; i.e., the system is linearly unstable in a reference frame moving with nonzero speed v towards higher numbered sites (we call this direction "downstream"). In the presence of noise, only a finite number of elements near the "upstream" edge are synchronized, evolving chaotically, while the remaining elements display complex incoherent time evolutions [see Fig. 1(a)]. If the boundary conditions are chosen to be periodic rather than open, the synchronized chaotic state becomes absolutely unstable

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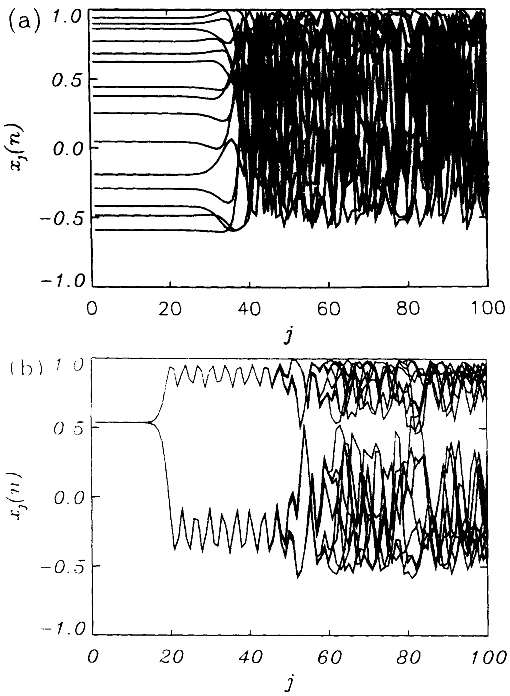

FIG. 1. Sixteen consecutive time steps in the eventual dynamics of an array with $\gamma_1=0.75$, $\gamma_2=0.05$, and $a^*=1.6$, at noise level $\sigma=10^{-10}$, are overlayed. The eventual dynamics without control (a) and with the control (b) of Eq. (3) applied at site 1 ($\delta a_{\max}=0.1$) are shown. All the site values at a given time are joined by solid lines, and were obtained from random initial conditions after $O(10^3)$ iterations of the array. The controlled array (b) exhibits regions of periods 1, 2, and 4.

[9], due to the reinjection of the amplified noise at the downstream boundary back into the system. In the discussion below, only arrays with open boundary conditions are considered.

It is important to remark that even though each individual element is operated at conditions well within the chaotic regime, the eventual motion in the presence of the control will be uniform in space and periodic in time. Consider the synchronization of an unstable period one orbit x^*, underlying the chaotic attractor at $a=a^*$. Without control, each element of the chain will behave chaotically. Suppose that site 1 takes on a value close to x^* at some time m; this is guaranteed to happen due to the ergodicity of the motion on a chaotic attractor. Assuming that the chaotic state has a coherence length of at least three sites (which holds for a wide range of parameters [9]), we expect $x_2(m)$ and $x_3(m)$ to take on values close to x^* as well. Applying a small parameter change $a=a^*+\delta a_1(m)$ at site 1 and at time m leads to the following linearized form for the evolution of site 2:

$$\delta x_2(m+1)=f'(x^*)[\gamma_0\delta x_2(m)+\gamma_1\delta x_1(m)+\gamma_2\delta x_3(m)] + A\gamma_1\delta a_1(m)\,, \tag{2}$$

where $\gamma_0\equiv 1-\gamma_1-\gamma_2$, $\delta x_j(m)\equiv x_j(m)-x^*$, the derivative of f is taken at $a=a^*$ and $A\equiv\partial f/\partial a$ at $x=x^*$ and $a=a^*$. Choosing $\delta a_1(m)$, the weak perturbation applied at time m, in such a way as to force element 2 to fall on the unstable fixed point at time $m+1$, yields the feedback law

$$\delta a_1(m)=-\frac{f'(x^*)}{A\gamma_1}[\gamma_0\delta x_2(m)+\gamma_1\delta x_1(m)+\gamma_2\delta x_3(m)]\,. \tag{3}$$

In cases where the dynamics of the individual elements cannot be described by a one-dimensional map, a feedback law which forces element 2 onto the local stable manifold of the unstable orbit must be formulated [10,11].

Will applying the above perturbations to site 1 from time m onwards force all the other elements to eventually synchronize? To answer this question, we perform a linear stability analysis of the uniform solution $x_j=x^*$ for all N sites, in the presence of the control law. Linearizing Eq. (1) around the uniform state at $a=a^*$ and using Eq. (3), we obtain

$$\begin{aligned}
&\delta x_1(n+1)=f'(x^*)/\gamma_1\{[\gamma_1\gamma_2-\gamma_0(1-\gamma_2)]\delta x_2(n)-(1-\gamma_2)\gamma_2\delta x_3(n)\}\,,\\
&\delta x_2(n+1)=0\,,\\
&\delta x_N(n+1)=f'(x^*)[(1-\gamma_1)\delta x_N(n)+\gamma_1\delta x_{N-1}(n)]\,,\\
&\delta x_j(n+1)=f'(x^*)[\gamma_0\delta x_j(n)+\gamma_1\delta x_{j-1}(n)+\gamma_2\delta x_{j+1}(n)]\,,
\end{aligned} \tag{4}$$

for $j=3,\ldots,N-1$. This system possesses a doubly degenerate eigenvalue at zero with the other $N-2$ eigenvalues given by $\lambda_q=f'(x^*)[\gamma_0+2\sqrt{\gamma_1\gamma_2}\cos\Theta_q]$ for $q=1,\ldots,N-2$, where $\Theta_q=\pi q/(N-1)$. Thus, the uniform state is linearly stable provided

$$|f'(x^*)|[(1-\gamma_1-\gamma_2)+2\sqrt{\gamma_1\gamma_2}]<1\,. \tag{5}$$

We expect that, in the absence of noise, a control signal applied at the upstream edge of a system obeying Eq. (5) will form a synchronizing front that is convected through the entire array. The introduction of even weak noise can destroy the synchronizing front in the presence of a convective instability. Such an instability is characterized by the existence of a frame moving with speed v, with respect to which the uniform state is linearly unstable; in other words, an initial disturbance $\delta x_i(0)$ is amplified as it is convected downstream according to the scaling regulation $\delta x_{i+vn}(n)\sim\delta x_i(0)e^{n\Lambda(v)}$, with $\Lambda(v)$ positive [12]. In Figs. 1(a) and 1(b), sixteen consecutive time steps in the eventual dynamics of an array at parameter values

$\gamma_1=0.75$, $\gamma_2=0.05$, and $a^*=1.6$ are shown. In order to make the effect of noise apparent, a random number uniformly distributed on $[-\sigma,\sigma]$ with $\sigma=10^{-10}$ was added to each site value at every time step of the evolution. In the absence of control [Fig. 1(a)], the first approximately thirty sites evolve chaotically, yet coherently, while sites further downstream are incoherent [9]. Some of the chaotic behavior is eliminated in Fig. 1(b) by applying feedback perturbations at site 1 according to Eq. (3), as long as their magnitudes are less than a prescribed $\delta a_{\max}=0.1$; otherwise the parameter is reset to its nominal value a^* until the chaotic trajectory returns to the neighborhood of x^*. Once the perturbations at site 1 stabilize element 2 onto x^*, a synchronizing front propagates downstream. Notice that the synchronizing front which stabilizes the unstable fixed point at $x^*=0.5376$ is destroyed downstream in favor of period 2 trajectories. Typically a region of several "spatial period doubling" [11] bifurcations appears, beyond which the dynamics of the elements becomes asynchronous and complex.

We now show how the "spatial bifurcations" in Fig. 1(b) can be eliminated in favor of the synchronized state, through the deployment of additional controllers which reinforce the synchronization front further along the chain. At a site located a distance ξ from site 1 (this distance is calculated below), a perturbation of the form specified in Eq. (3) is applied, where the site indices are translated by ξ elements. As before, these perturbations are applied only as long as they are smaller than the prescribed $\delta a_{\max}$. This procedure is reiterated for additional sites, each located ξ elements from the previous control site, until the boundary of the array is reached. In Fig. 2, the advance of the synchronizing front in a 200 site array with $\xi=14$ is shown for every 1000 iterates, starting from random initial conditions. All parameters of the array and controllers are identical to those used in Fig. 1. The synchronizing front moves monotonically to the right with a speed that evolves in a periodic fashion;

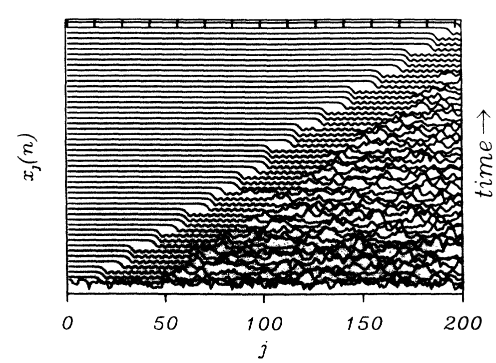

FIG. 2. Evolution of the synchronizing front in the system of Fig. 1, with controllers spaced every 14 sites (tick marks at the top of the figure indicate their positions) with $\delta a_{\max}=0.1$ and $\sigma=10^{-10}$. The state of the array is shown every 1000 iterates (site values are joined by solid lines), starting from random initial conditions.

each time the front reaches a new control site, it hovers around for a long time relative to the time it takes it to advance between control sites.

How is the distance ξ between control sites determined? To answer this question, consider the effect of adding a δ-correlated random variable distributed on $[-\sigma,\sigma]$ to the right hand sides of the noise-free dynamics of Eq. (1). An initial disturbance of strength σ introduced at site 1 of a uniformly synchronized state will grow to have magnitude $\sigma e^{\Lambda(v)l/v}$ at site l at time l/v later. Assuming noise is continuously injected into the system, the maximal disturbance felt at site l is that corresponding to a convective instability moving with a speed $v=v^*$ for which $\Lambda(v)/v$ is maximal, so that v^* is the solution of the equation $d/dv[\Lambda(v)]=\Lambda(v)/v$. Taking the uniform state as an initial condition, it can be seen that a control of the form of Eq. (3), applied at site l at times proceeding l/v^*, will manage to reduce the disturbance down to $O(\sigma)$, provided $\delta a_{\max}>C\sigma e^{l\Lambda(v^*)/v^*}$, where C is a constant which depends on the gain coefficients of the control law in Eq. (3). Thus, the maximal distance ξ between controllers which renders the control procedure effective is

$$\xi\sim\frac{1}{\Lambda'(v^*)}\ln(\delta a_{\max}/C\sigma)\,, \tag{6}$$

where $\Lambda'(v)$ denotes the derivative of $\Lambda(v)$.

The above scaling relation was verified for unidirectional coupling ($\gamma_2=0$) by determining ξ numerically for several noise levels and perturbation strengths at fixed values of γ_1 and a^*. The deviations from the scaling of Eq. (6) occur only when nonlinear effects become significant, i.e., when $\delta a_{\max}\gg\sigma^{1/2}$. Each determination of ξ involved implementing control procedures with increasing values for the intercontroller spacing l. ξ was determined as the largest spacing l for which it was possible to control a system consisting of $N=kl$ sites, where we chose $k=10$. For $l>\xi$, the synchronizing front may never make it over to the Nth site; even if it does happen to move through the entire system, the front quickly recedes upstream and typically fluctuates wildly back and forth without attaining a steady state. Although the values of ξ were calculated numerically for finite k, we believe that they represent values in the thermodynamic limit ($k\to\infty$). Indeed, no deterioration in the steady state is observed near control sites which are further downstream. We note that in the calculations the specified value of $\delta a_{\max}$ was used for all control sites except the one at site 1, for which a larger value of $\delta a_{\max}=0.1$ was used. In this way, the problem of waiting an insurmountable amount of time for the control to be initiated is overcome. Although realistic noise levels are orders of magnitude larger than those examined in the model ($\sigma<10^{-7}$), large intercontroller spacings will be effective in experiments, provided the quantity $\Lambda'(v^*)$ is not too large. Notice from Eq. (6) that the dependence of ξ on this quantity is linear as opposed to the logarithmic dependence on noise.

Synchronizing arrays with controllers separated by the distance ξ of Eq. (6) can be more complicated in cases of bidirectional coupling ($\gamma_2 \neq 0$). For such arrays, a front propagates in the upstream direction, in addition to the synchronizing front which moves downstream. Unlike the downstream flow, disturbances decay as they propagate upstream. Thus, even though an unsynchronized element forms a disturbance which decays upstream, it may still have a disabling effect on the reinforcement upstream. In other words, although the synchronized state is linearly stable for all moving frames at separation ξ between controllers, it is not necessarily a global attractor. The method we use to enter the small basin of attraction is to initially employ additional controllers separated by a smaller distance, say $\xi/2$, in order to synchronize the array. Then, perturbations at every second control site are turned off (see [11] for details). Using this technique, the scaling relation of Eq. (6) has also been verified numerically for arrays with $\gamma_2 \neq 0$.

The synchronized steady state obtained using intercontroller spacing ξ can be deduced from Eq. (6) to be enclosed in an envelope $g(y)$, given by $\ln g(y) = \ln\sigma + \Lambda'(v^*)(y \bmod \xi)$, where y is the coordinate along the array. The value of v^* can be calculated from the spectrum $\Lambda(v)$ of spatial instabilities of the linearized system (1). For unidirectional coupling ($\gamma_2 = 0$), a disturbance $\delta x_2(0)$ grows as it propagates down the chain with speed v according to

$$\Lambda(v) = \ln|f'(x^*)| - (1-v)\ln\frac{1-v}{1-\gamma_1} - v\ln\frac{v}{\gamma_1}$$

(see Refs. [2,12]), leading to $v^* = 1 - |f'(x^*)|(1-\gamma_1)$. The numerically obtained steady state for a system with $\gamma_1 = 0.65$, $\gamma_2 = 0$, $a^* = 1.6$, and $\sigma = 10^{-8}$ is shown in Fig. 3 as circles. Close agreement is found with the steady state envelope $g(y)$ (plotted as a solid line), which was obtained using $\xi = 13$ and the expression for v^*.

We now discuss the relevance of our control to experimental systems, in which making many simultaneous measurements is often unachievable. The great advantage of the control method presented in this paper is that it is inherently local in nature, involving only three measurements in a spatial neighborhood of the site at which a perturbation is applied. The number of required measurements may be reduced even further by utilizing knowledge of the shape of the envelope $g(y)$, in order to deduce the value at a particular site from that at an adjacent site. Another way in which the number of measurements can be reduced is to replace the local linear dynamical model of Eq. (4) by one which depends only on a *single* scalar variable and the history of the applied parameter perturbations as was introduced in Ref. [10].

To summarize, we have shown how a convectively unstable spatially extended system consisting of locally coupled chaotic elements can be controlled to behave periodically by the application of small feedback perturbations

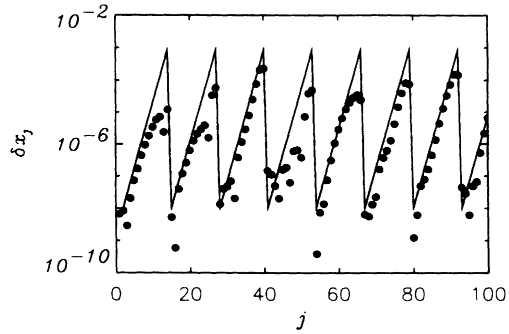

FIG. 3. Log plot of the steady state shape of the synchronized array (circles) determined numerically for a system with $\gamma_1 = 0.65$, $\gamma_2 = 0$, $a^* = 1.6$, $\xi = 13$, $\sigma = 10^{-8}$, and $\delta a_{\max} = 0.01$. The front is compared with the steady state envelope $g(y)$ (solid line).

to an external parameter at several spatial locations. Although the above discussion was limited to synchronizing a fixed point, the method is easily extended to stabilizing higher periodic orbits. Associated with each element of a period k orbit is a synchronizing front, formed by the consecutive application of k *different* control laws of the type in Eq. (3). Because of the relative ease with which the control scheme presented here can be implemented, we anticipate experimental applications to be forthcoming.

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