Proof of Page's Conjecture on the Average Entropy of a Subsystem

S. K. Foong* and S. Kanno
Department of Physics, Ibaraki University, Mito 310, Japan
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Page conjectured very recently [Phys. Rev. Lett. **71**, 1291 (1993)] that if a quantum system of Hilbert space dimension mn is in a random pure state, the average entropy of a subsystem of dimension $m \leq n$ is given by $S_{m,n} = \sum_{k=n+1}^{mn} \frac{1}{k} - \frac{m-1}{2n}$. We outline a proof of this elegant formula.

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Very recently Page [1] considered the problem of getting entropy out of a system in a pure quantum state. He took a quantum system AB of Hilbert dimension mn with normalized density matrix, and divided it into two coupled subsystems, A and B, of dimension m and n, respectively, with $m \leq n$.

Suppose the system AB is in a pure state; then its entropy $S_{AB} = 0$. However, the entropy of the subsystems S_A and S_B are positive. Page and several other authors, as cited in his Letter, were interested in $S_{m,n}$, the average of the entropy S_A over all pure states of the total system. The pure states were assumed to have the unitarily invariant Haar measure on the space of unit vectors $|\psi\rangle$ in the mn-dimensional Hilbert space of the total system. Page conjectured an elegant formula for $S_{m,n}$ based on its form for $m = 2, 3, 4$, and 5. He also obtained an approximate formula valid for large m and n, starting from the equivalent of the following exact formula,

$$S_{m,n} \equiv \langle S_A \rangle = -N(m,n) \int \sum_{i=1}^{m} p_i \ln p_i \; P(p_1, \ldots, p_m) \prod_{i=1}^{m} dp_i, \tag{1}$$

where

$$P(p_1, \ldots, p_m) \prod_{i=1}^{m} dp_i = \delta\Big(1 - \sum_{i=1}^{m} p_i\Big) \prod_{1 \leq i < j \leq m} (p_i - p_j)^2 \prod_{k=1}^{m} (p_k^{n-m} dp_k) \tag{2}$$

which differs from that of Page by the normalization factor

$$N(m,n) = \left[\int P(p_1, \ldots, p_m) \prod_{i=1}^{m} dp_i \right]^{-1}. \tag{3}$$

We shall assume these formulas and proceed to prove the conjecture by evaluating the integrals exactly. Because of limitation of space, the details [2] will be published elsewhere. To begin, the integrand is multiplied by a damping factor $\exp(-\sum_{i=1}^{m} \epsilon_i p_i)$ which serves two advantages: (1) the p_i in the products $\prod$ may be replaced by $D_i = -\frac{\partial}{\partial \epsilon_i}$ and (2) if the delta function is expressed as an integral, the order of integration can be interchanged such that the integrations over p_i are done first. We have consequently

$$N(m,n)^{-1} = \lim_{\epsilon \to 0} \left[\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega} \prod_{1 \leq i < j \leq m} (D_i - D_j)^2 \prod_{l=1}^{m} D_l^{n-m} \prod_{l=1}^{m} \frac{1}{\epsilon_k'} \right]_{\epsilon_i = \epsilon}, \tag{4}$$

where $\epsilon_k' \equiv \epsilon_k + i\omega$, and the problem is transformed from one of evaluating multidimensional integrals to one of evaluating partial differentiations, and just one integration. To save writing, we shall drop the prime in the rest of the paper.

Similarly, the other factor on the right of Eq. (1) can be reexpressed as

$$- \int \sum_{i=1}^{m} p_i \ln p_i \; P(p_1, \ldots, p_m) \prod_{i=1}^{m} dp_i$$

$$= \lim_{\epsilon \to 0} \left\{ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega} \prod_{1 \leq i < j \leq m} (D_i - D_j)^2 \prod_{l=1}^{m} D_l^{n-m} \sum_{l=1}^{m} D_l \left[(\gamma + \ln \epsilon_l) \prod_{j=1}^{m} \frac{1}{\epsilon_j} \right] \right\}_{\epsilon_i = \epsilon}, \tag{5}$$

where γ is Euler's constant.

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We consider first the evaluation of $N(m,n)$. This is facilitated by noting [3]

$$\prod_{1\le i<j\le m}(D_i-D_j)=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ D_1 & D_2 & \cdots & D_m \\ \vdots & & \cdots & \vdots \\ D_1^{m-1} & D_2^{m-1} & \cdots & D_m^{m-1} \end{vmatrix}. \tag{6}$$

The matrix will be denoted as $\mathcal{D}$, i.e., the determinant $|\mathcal{D}|$, and hence

$$\prod_{1\le i<j\le m}(D_i-D_j)^2=|\mathcal{D}|^2=|\mathcal{D}||\mathcal{D}^T|=|\mathcal{D}\mathcal{D}^T|. \tag{7}$$

Observe that in both Eq. (4) and Eq. (5) $|\mathcal{D}\mathcal{D}^T|$ acts on a symmetric function $f(\{\epsilon\})$ of ϵ_i ($i=1,\ldots,m$), and the result is evaluated at $\epsilon_i=\epsilon$. It turns out for such a case, the following simplification can be established [2]:

$$|\mathcal{D}\mathcal{D}^T|\; f(\{\epsilon_i\})|_{\epsilon_i=\epsilon}=m!\;|\mathcal{D}|\;D_2D_3^2\cdots D_m^{m-1}f(\{\epsilon_i\})|_{\epsilon_i=\epsilon}. \tag{8}$$

Set $n-m+1=\alpha$. For the symmetric function in Eq. (4), we adopt the notation

$$\prod_{l=1}^{m}D_l^{n-m}\prod_{l=1}^{m}\frac{1}{\epsilon_k'}\equiv f_1(\{\epsilon_i\}). \tag{9}$$

Then, we have

$$|\mathcal{D}\mathcal{D}^T|\; f_1(\{\epsilon_i\})|_{\epsilon_i=\epsilon}=m!|\mathcal{D}|\;D_1^{\alpha-1}D_2^{\alpha}\cdots D_m^{\alpha+m-2}\prod_{k=1}^{m}\frac{1}{\epsilon_k'}\bigg|_{\epsilon_i=\epsilon}=m!\prod_{k=0}^{m-1}(\alpha-1+k)!|\mathcal{D}|\prod_{k=1}^{m}\frac{1}{\epsilon_k^{\alpha+k-1}}\bigg|_{\epsilon_i=\epsilon}. \tag{10}$$

Observe that

$$|\mathcal{D}|\prod_{k=1}^{m}\frac{1}{\epsilon_k^{\alpha+k-1}}\bigg|_{\epsilon_i=\epsilon}=\frac{1}{\epsilon^{m(\alpha+m-1)}}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha & \alpha+1 & \cdots & \alpha+m-1 \\ \vdots & \vdots & \cdots & \vdots \\ \prod_{k=0}^{m-2}(\alpha+k) & \prod_{k=1}^{m-1}(\alpha+k) & \cdots & \prod_{k=m-1}^{2m-3}(\alpha+k) \end{vmatrix}. \tag{11}$$

The determinant can be evaluated to be $\prod_{k=1}^{m-1}k!$ by a sequence of two operations where at the lth step (1) take the jth ($j\ge l+1$) column and subtract from it the ($j-1$)th, and the resulting column replaces the jth column and (2) factor the constant in front of every $i\ge l+1$ row yielding the factorial $(m-l)!$.

Collecting the results above, we have

$$|\mathcal{D}\mathcal{D}^T|\; f_1(\{\epsilon_i\})|_{\epsilon_i=\epsilon}=m!N(\alpha,m,\epsilon), \tag{12}$$

where

$$N(\alpha,m,\epsilon)\equiv\frac{1}{\epsilon^{m(\alpha+m-1)}}\prod_{k=0}^{m-1}k!(\alpha-1+k)!. \tag{13}$$

With Eq. (12), the integral over ω in Eq. (4) then yields a factor $1/\Gamma(m(\alpha+m-1))$, and consequently the normalization is given by

$$N(m,n)=(mn-1)!\left\{\prod_{k=1}^{m}\left[k!(n-k)!\right]\right\}^{-1}. \tag{14}$$

We now proceed to evaluate the right-hand side (rhs) of Eq. (5). Let

$$f_2(\{\epsilon_i\})\equiv\prod_{k=1}^{m}D_k^{\alpha-1}\sum_{l=1}^{m}D_l\left[(\gamma+\ln\epsilon_l)\prod_{j=1}^{m}\frac{1}{\epsilon_j}\right], \tag{15}$$

and we shall first evaluate

$$|\mathcal{D}\mathcal{D}^T|\ f_2(\{\epsilon_i\})|_{\epsilon_i=\epsilon} = m!|\mathcal{D}|\ D_1^{\alpha-1}D_2^{\alpha}\cdots D_m^{\alpha+m-2}\sum_{l=1}^{m} D_l\left[(\gamma+\ln\epsilon_l)\prod_{j=1}^{m}\frac{1}{\epsilon_j}\right]. \tag{16}$$

The $\ln\epsilon_l$ in Eq. (16) makes the calculation more complicated. For convenience, we record the following formula:

$$D^n\left(\frac{(\gamma+\ln\epsilon)}{\epsilon^{\alpha+1}}\right) = \frac{1}{\epsilon^{\alpha+n+1}}\left[(\gamma+\ln\epsilon)\prod_{i=1}^{n}(\alpha+i) - f_n(\alpha)\right], \tag{17}$$

where the first term in $[\cdots]$ is the result of the numerator $(\gamma+\ln\epsilon)$ never having been differentiated, whereas the second term f_n corresponds to it having been differentiated at least once, and it is given by

$$f_n(\alpha) = n!\sum_{k=1}^{n-1}\frac{\prod_{i=1}^{k}(\alpha+i)}{k!(n-k)} + (n-1)!\ , \tag{18}$$

with $f_0=0$ and $f_1=1$. This helps in the evaluation of the following factor appearing in the rhs of Eq. (16):

$$D_1^{\alpha-1}D_2^{\alpha}\cdots D_l^{\alpha+l-1}\cdots D_m^{\alpha+m-2}\left[(\gamma+\ln\epsilon_l)\prod_{j=1}^{m}\frac{1}{\epsilon_j}\right]$$

$$= (\alpha+l-1)\prod_{k=0}^{m-1}(\alpha-1+k)!\left\{\frac{1}{\epsilon_1^{\alpha}}\frac{1}{\epsilon_2^{\alpha+1}}\cdots\frac{1}{\epsilon_{l-1}^{\alpha+l-2}}\frac{[(\gamma_l'+\ln\epsilon_l)]}{\epsilon_l^{\alpha+l}}\frac{1}{\epsilon_{l+1}^{\alpha+l}}\cdots\frac{1}{\epsilon_m^{\alpha+m-1}}\right\}, \tag{19}$$

where $\gamma_l' \equiv \gamma - \sum_{k=1}^{\alpha+l-1}\frac{1}{k}$. Notice that the exponent of ϵ_l and ϵ_{l+1} are equal on the right of Eq. (19); therefore when acted on by $\mathcal{D}$ the term proportional to γ_l' vanishes because the two columns, lth and $(l+1)$th, in the resulting determinant are equal when ϵ_i are set to equal ϵ. This is true for $l<m$, but not for $l=m$—there is no column to the right of the mth, and this particular case will be considered following Eq. (21). By the same reasoning, the term corresponding to that where the logarithmic term never having been differentiated [recall the remark following Eq. (17)] also vanishes, and consequently the result of $\mathcal{D}$ acting on the factor within { } in Eq. (19) is the same as the rhs of Eq. (11) except the lth column is replaced by $-(f_0,f_1,\ldots,f_{m-1})^T$, and the exponent of ϵ is increased by 1 due to the one extra differentiation D_l. Upon evaluation of this nontrivial looking determinant, we obtain the contribution corresponding to the lth term as

$$m!\frac{l(\alpha+l-1)}{m-l}\frac{N(\alpha,m,\epsilon)}{\epsilon}. \tag{20}$$

Summing over the term from $l=1$ to $l=m-1$, their contribution to $|\mathcal{D}\mathcal{D}^T|f_2$ is, with $\alpha=n-m+1$, given by

$$m!\left[mn\sum_{k=1}^{m-1}\frac{1}{k} - \left(n+\frac{m}{2}\right)(m-1)\right]\frac{N(\alpha,m,\epsilon)}{\epsilon}. \tag{21}$$

For the $l=m$ term, we cannot leave out the terms containing γ_m' nor the logarithm, and its contribution is given by

$$(\gamma'_m+\ln\epsilon)\begin{vmatrix} 1 & \cdots & 1 & 1 \\ \alpha & \cdots & \alpha+m-2 & \alpha+m \\ \vdots & \cdots & \vdots & \vdots \\ \prod_{k=0}^{m-2}(\alpha+k) & \cdots & \prod_{k=m-2}^{2m-4}(\alpha+k) & \prod_{k=m}^{2m-2}(\alpha+k) \end{vmatrix}$$

$$-\begin{vmatrix} 1 & \cdots & 1 & f_0 \\ \alpha & \cdots & \alpha+m-2 & f_1 \\ \vdots & \cdots & \vdots & \vdots \\ \prod_{k=0}^{m-2}(\alpha+k) & \cdots & \prod_{k=m-2}^{2m-4}(\alpha+k) & f_{m-1} \end{vmatrix} \tag{22}$$

multiplied by $(\alpha+m-1)\prod_{k=0}^{m-1}(\alpha-1+k)!/\epsilon^{m(\alpha+m-1)+1}$. The determinants above are evaluated to be

$$m\prod_{k=1}^{m-1}k! \quad \text{and} \quad \left[m\sum_{k=1}^{m-1}\frac{1}{k}-(m-1)\right]\prod_{k=1}^{m-1}k!, \tag{23}$$

respectively. Substituting α by $n-m+1$, the contribution due to the $l=m$ term is

$$m!n\left\{m(\gamma'_m+\ln\epsilon)-\left[m\sum_{k=1}^{m-1}\frac{1}{k}-(m-1)\right]\right\}\frac{N(\alpha,m,\epsilon)}{\epsilon}. \tag{24}$$

We have finally finished doing the differentiations, and adding Eq. (21) and expression (24) gives

$$|\mathcal{D}\mathcal{D}^T|f_2(\{\epsilon_i\})|_{\epsilon_i=\epsilon}=m!\prod_{k=0}^{m-1}(n-m+k)!k!\frac{(\eta+mn\ln\epsilon)}{\epsilon^{mn+1}}, \tag{25}$$

where $\eta=mn\gamma'-\frac{m}{2}(m-1)$.

For the integration over ω, we need the following formula:

$$I_n\equiv\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega}\frac{\ln(\epsilon+i\omega)}{(\epsilon+i\omega)^n}=\frac{e^{-\epsilon}}{(n-1)!}\left(\sum_{k=1}^{n-1}\frac{1}{k}-\gamma\right), \tag{26}$$

which can be established from the recurrence relation

$$I_n=\frac{I_{n-1}}{n-1}+\frac{1}{(n-1)\Gamma(n)}, \tag{27}$$

with $I_1=-\gamma$ obtained by contour integration.

Finally, we obtain

$$-\int\sum_{i=1}^{m}p_i\ln p_i\;P(p_1,\ldots,p_m)\prod_{i=1}^{m}dp_i=\frac{m!}{(mn)!}\prod_{k=0}^{m-1}(n-m+k)!k!\left[mn\sum_{k=n+1}^{mn}\frac{1}{k}-\frac{m}{2}(m-1)\right], \tag{28}$$

which when multiplied by $N(m,n)$ Eq. (14) yields

$$S_{m,n}=\sum_{k=n+1}^{mn}\frac{1}{k}-\frac{m-1}{2n}, \tag{29}$$

Page's conjecture.

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* Electronic address: foong@tansei.cc.u-tokyo.ac.jp

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[2] S. Kanno and S. K. Foong (to be published).
[3] This interesting relation was found after a couple of false starts, but before finding out it is given as problem No. 35 in Frank Ayres, Jr., *Schaum's Outline of Theory and Problems of Matrices* (Schaum Publishing Co., New York, 1962), p. 31; also available in Asian edition (McGraw-Hill International, Singapore, 1982).