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Complex Wave-Field Reconstruction Using Phase-Space Tomography

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A method is proposed for the experimental determination of the amplitude and phase structure of a quasimonochromatic wave field in a plane normal to its propagation direction. The wave field may represent either a scalar electromagnetic (EM) field or the quantum mechanical (QM) wave function of a matter wave. For coherent EM fields or pure QM states, the method uniquely reconstructs the complex wave fields. For partially coherent EM fields or mixed QM states, it reconstructs the two-point correlation function or density matrix. The experiment uses only intensity measurements and refractive optics (lenses), and the data analysis algorithm is noniterative and requires no deconvolution.

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The complete characterization of a fully coherent, quasimonochromatic, classical electromagnetic (EM) wave or a quantum-mechanical (QM) matter-wave field requires the specification of its amplitude and phase as functions of position, or equivalently its complex amplitude $\psi(\mathbf{r})$. Here $\psi(\mathbf{r})$ refers to either the wave function (in the case of a matter wave) or the electric field (in the case of a scalar EM wave). In the case of partial coherence, where an ensemble of similarly prepared systems is under consideration, a useful characterization is provided in the EM case by the two-point correlation function, also called the mutual intensity, which is equal to

$$\Gamma(\mathbf{r},\mathbf{r}') = \langle \psi(\mathbf{r}) \psi^*(\mathbf{r}') \rangle , \tag{1}$$

where the brackets indicate an ensemble average over the set of realizations of the functions $\psi(\mathbf{r})$. In the case of a field obeying Gaussian statistics this function provides its complete statistical characterization [1]. In the QM case, the analogous quantity is the density matrix, also given by Eq. (1) [2]. It provides a complete description for any one-particle state (ignoring spin). The frequency label on $\Gamma(\mathbf{r},\mathbf{r}')$ has been suppressed since we are considering quasimonochromatic fields.

There has been a long-standing interest in the EM problem of phase retrieval [3]. Earlier approaches relied on interferometric techniques, and often an iterative computational scheme to find a best fit, but not necessarily unique, form of the field. Recently, for the case of a coherent field, noniterative techniques were demonstrated for determining the field [4,5]. These techniques also make no recourse to interferometry, using instead diffractive optics [5], or, more interestingly in our context, only refractive optics [4]. Such methods provide a unique determination of a coherent field, to within spatial bandwidth limits imposed by the measurement scheme. The case of a partially coherent field is not so simple, and no general solution has been previously reported [4,6,7].

In the QM case, recent advances in matter-wave interferometry [8] have made it important to have methods for experimentally determining the spatial coherence properties of the matter waves, i.e., the density matrix, especially in the case of particles described by a mixed state (partially coherent matter wave) rather than a pure

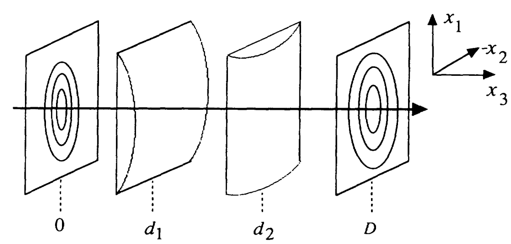

FIG. 1. Wave field in the $x_3=0$ plane propagates through two cylindrical lenses, oriented along the x_1,x_2 directions, and is detected in a plane at $x_3=D$. Intensity data are collected for many combinations of focal lengths f_1,f_2 and distances d_1,d_2.

state.

In this Letter, a noniterative method, shown in Fig. 1, is presented for experimental determination of the correlation function and/or the density matrix $\Gamma(\mathbf{r},\mathbf{r}')$ of a wave field with arbitrary state of coherence, either EM or QM, from "intensity" measurements only, without recourse to either interferometry or diffractive optics [9]. The method determines $\Gamma(\mathbf{r},\mathbf{r}')$ in a two-dimensional (2D) plane, which is not obscured from view by any medium. This method uses only refractive optics (lenses) and is limited only by spatial resolution. By "intensity" is meant the quantity

$$I(\mathbf{r})=\Gamma(\mathbf{r},\mathbf{r})=\langle|\psi(\mathbf{r})|^2\rangle, \qquad (2)$$

which in the EM case is proportional to the ensembled-averaged optical field intensity, and in the QM case is the particle-position probability density.

The method relies on the transformation properties of the Wigner function $W(\mathbf{r},\mathbf{k})$ corresponding to the state (EM or QM) of the wave field defined in the $x_3=0$ plane [and using $\mathbf{r}=(x_1,x_2,x_3)$] as [10]

$$W(x_1,k_1,x_2,k_2)=(1/\pi^2)\int_{-\infty}^{\infty}dx_1'\int_{-\infty}^{\infty}dx_2'\exp(-i2k_1x_1'-i2k_2x_2')\langle\psi(x_1+x_1',x_2+x_2',0)\psi^*(x_1-x_1',x_2-x_2',0)\rangle. \qquad (3)$$

It is assumed that the wave travels nominally in the $+x_3$ direction and that the transverse (x_1,x_2) structure of the wave is smooth on the scale of many wavelengths; thus the paraxial (small-angle) approximation is applicable [1]. In the QM case, k_1 and k_2 are interpreted as transverse momenta (with $\hbar=1$), while in the EM case they are transverse propagation constants proportional to spatial frequencies. In both cases we scale x_i and k_i by a common length x_0 so they are dimensionless. Note that Eq. (3) provides a unique correspondence between $\Gamma(x_1,x_2,0,x_1',x_2',0)$ and $W(x_1,k_1,x_2,k_2)$.

The proposed method uses a set of measured values of intensity $I(\mathbf{r})$ to reconstruct uniquely, by tomographic inversion, the Wigner function W, and thus Γ in the $x_3=0$ plane. In this regard the approach is similar in spirit to that discussed by Nugent [6]. That approach was subsequently shown by Hazak [7] to be incomplete due to an inability in the proposed scheme to cover enough of the four-dimensional (4D) phase space in which W is defined to allow reconstruction. The present discussion shows how to overcome that limitation by using lenses. First the necessary measurements will be described, followed by the data analysis technique.

Consider the wave traveling in the x_3 direction and being normally incident on a pair of thin, cylindrical lenses. One lens, located at $x_3=d_1$ has focal length f_1 in the x_1 direction, and the other lens located at $x_3=d_2$ has focal length f_2 in the x_2 direction. (All lengths are scaled by x_0.) Experimentally it is only necessary to measure the intensity distribution $I(x_1,x_2,x_3)$ in various planes located at $x_3=D$, where $D>d_1,d_2$, for various combinations of discrete values of focal lengths f_1,f_2 and/or distances d_1,d_2. Such a measurement is simple in the EM case and is presently feasible also in the QM particle case, using atom optics. By proper analysis of the intensity data, the function Γ in the $x_3=0$ plane can be reconstructed uniquely, within spatial bandwidth limits, by a noniterative, linear transform of the data.

To see this, consider that within the paraxial approximation, the effect of the x_1 direction cylindrical lens at $x_3=d_1$ is (ideally) to multiply the incident wave $\psi(x_1,x_2,d_1)$ by a transversely varying phase factor $\exp(ik_3x_1^2/2f_1)$, where k_3 is the scaled, dimensionless momentum (or propagation constant) in the x_3 direction, and is presumed known. The x_2-direction lens has a similar effect. The combined effect of the lenses and propagation is described by a Fresnel integral over the $x_3=0$ plane, giving for the field at $\mathbf{r}=(x_1,x_2,x_3=D)$ [1]

$$\psi(\mathbf{r})=\int_{-\infty}^{\infty}dx_1'\int_{-\infty}^{\infty}dx_2'K(\mathbf{r},x_1',x_2')\psi(x_1',x_2',0), \qquad (4a)$$

where

$$K(\mathbf{r},x_1',x_2')=C\exp[ih(x_1,x_2)]\exp\left[ik_3\left(\frac{x_1x_1'}{L_1}+\frac{x_2x_2'}{L_2}-\frac{(x_1')^2}{2R_1}-\frac{(x_2')^2}{2R_2}\right)\right], \qquad (4b)$$

and the curvature radii R_1 and R_2 are given by $R_i=R_{i0}+d_i$, where

$$\frac{1}{R_{i0}}=\frac{1}{D-d_i}-\frac{1}{f_i} \quad (i=1,2), \qquad (4c)$$

and the effective propagation lengths are given by

$$L_i=(D-d_i)(1+d_i/R_{i0}). \qquad (4d)$$

In Eq. (4b) C is a complex constant for a given lens configuration and $h(x_1,x_2)$ is a phase, whose precise forms are unimportant here. It is convenient to assume that $\psi(x_1,x_2,0)$ has been multiplied by a constant (related to the total flux) to make it normalized according to

$$\int\int|\psi(x_1,x_2,0)|^2dx_1dx_2=1;$$

then it can be shown that this normalization is upheld for all x_3. Equation (4a) has been called a fractional Fourier transform, since it interpolates between $\psi(x_1,x_2,0)$ at $d_i=D=0$ and its (2D) Fourier transform at $d_i=D/2=f_1=f_2$ [11,12].

Next we will consider how the Wigner function transforms under the influence of the lenses, and show that it corresponds to a rotation in phase space (as pointed out for spherical lenses by Lohmann [12]). We then show that this allows projection tomography to be carried out

[13]. Consider the 4D phase space with (dimensionless) coordinates x_1,k_1,x_2,k_2 in which $W(x_1,k_1,x_2,k_2)$ is defined. Define new coordinates obtained by formally rotating by angles θ and β,

$$x_{1\theta}=x_1\cos\theta+k_1\sin\theta,\quad k_{1\theta}=-x_1\sin\theta+k_1\cos\theta\,,$$
$$x_{2\beta}=x_2\cos\beta+k_2\sin\beta,\quad k_{2\beta}=-x_2\sin\beta+k_2\cos\beta\,. \tag{5}$$

If projection integrals are carried out over the Wigner function along lines parallel to the $k_{1\theta}$ axis, for fixed values of $x_{1\theta}$, one obtains the projected distributions

$$P_\theta(x_{1\theta};x_2,k_2)=\int_{-\infty}^{\infty}W(x_1,k_1,x_2,k_2)dk_{1\theta}\,, \tag{6}$$

each distinguished by a different value of θ. It is understood that in Eq. (6) the arguments of W are to be expressed in terms of the rotated variables. The basis of projection tomography is that if one can measure the projected function $P_\theta(x_{1\theta};x_2,k_2)$ for a sufficient set of values of θ over a π interval, then Eq. (6) can be inverted to obtain $W(x_1,k_1,x_2,k_2)$ by using the inverse Radon transform [14]. We have demonstrated experimentally such a phase-space method for reconstructing the Wigner function for a quantum optical system having a single degree of freedom [15], following an earlier theoretical proposal [16]. A time-frequency domain reconstruction has also been discussed [17].

In the present case there are 2 degrees of freedom, and we need to perform another set of projections along lines parallel to the $k_{2\beta}$ axis, which yield the distributions

$$P_{\theta\beta}(x_{1\theta},x_{2\beta})=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W(x_1,k_1,x_2,k_2)dk_{1\theta}dk_{2\beta}\,. \tag{7}$$

Thus, if we can collect enough joint distributions $P_{\theta\beta}(x_{1\theta},x_{2\beta})$ for many combinations of θ,β values, we can apply successive inversions to Eq. (7) (first in x_2,k_2 then in x_1,k_1) to obtain W.

It now remains to prove that the Fresnel-integral transformation Eq. (4) actually affects the double projection integral of the Wigner function in Eq. (7). Simply insert Eq. (3) into Eq. (7) and change integration variables —this yields for the projected distributions

$$P_{\theta\beta}(x_{1\theta},x_{2\beta})=\langle|\psi_{\theta\beta}(x_{1\theta},x_{2\beta})|^2\rangle\,, \tag{8}$$

where the transformed wave field is

$$\psi_{\theta\beta}(x_{1\theta},x_{2\beta})=\int_{-\infty}^{\infty}dx_1'\int_{-\infty}^{\infty}dx_2'K_{\theta\beta}(x_{1\theta},x_{2\beta},x_1',x_2')\psi(x_1',x_2',0)\,, \tag{9a}$$

where

$$K_{\theta\beta}(x_{1\theta},x_{2\beta},x_1',x_2')=C_{\theta\beta}\exp[ig(x_{1\theta},x_{2\beta})]\exp\{-i[x_{1\theta}x_1'\csc\theta-\tfrac{1}{2}(x_1')^2\cot\theta]\}$$
$$\times\exp\{-i[x_{2\beta}x_2'\csc\beta-\tfrac{1}{2}(x_2')^2\cot\beta]\}\,, \tag{9b}$$

where $C_{\theta\beta}=(4\pi^2|\sin\theta\sin\beta|)^{-1/2}$ and $g(x_{1\theta},x_{2\beta})$ is an unimportant phase. Equation (8) is of the form of an intensity, defined in Eq. (2). In fact, it is equal to the intensity $I(x_1,x_2,D)$ of the field after the cylindrical lenses, given by Eq. (4). This equivalence can be established by rescaling the variables:

$$\tilde{x}_{1\theta}=-x_{1\theta}(L_1\csc\theta/k_3)\,,$$
$$\tilde{x}_{2\beta}=-x_{2\beta}(L_2\csc\beta/k_3)\,, \tag{10}$$

and choosing the correspondences

$$k_3/R_1=-\cot\theta,\quad k_3/R_2=-\cot\beta\,. \tag{11}$$

With these definitions $\psi_{\theta\beta}(x_{1\theta},x_{2\beta})$ in Eq. (9) is equal to $\psi(x_1,x_2,D)$ in Eq. (4) to within a phase factor which is not important when considering only the intensity. Thus varying R_1 (R_2) corresponds to rotation of the x_1,k_1 (x_2,k_2) coordinate system by angle θ (β). For tomographic inversion it is sufficient to vary θ and β in N equal increments between zero and $\pi-\pi/N$. The number of different angles needed depends on the complexity of the structure of the Wigner function. These rotations can be accomplished independently by varying the focal lengths f_1,f_2 and/or distances d_1,d_2 independently. This independence of rotation of the x_1,x_2 dimensions, necessary for tomographic inversion, was missing in the proposal by Nugent [6].

In order to demonstrate that the data analysis scheme is feasible, we have numerically simulated the reconstruction of a coherent wave field with the following form:

$$\psi(x_1,x_2,0)\propto\exp[iax_2^2]$$
$$\times\exp[-(x_1+x_2)^2\delta/4-(x_1-x_2)^2/4\delta]\,. \tag{12}$$

This function is plotted in Fig. 2, which shows both the amplitude and phase structure. From this function we analytically determined the Wigner function using Eq. (3), and thereby the projected distributions defined in Eq. (7). In this case we used 21 values of each angle θ,β, and 32 values of each variable $x_{1\theta},x_{2\beta}$ to simulate experimental histogram. We treated these distributions numerically as if they were experimentally measured data files, and reconstructed the Wigner function using the filtered back-projection algorithm for the inverse Radon transform [14]. Equation (7) had to be inverted $21\times32=672$ times, and then Eq. (6) had to be inverted $(32)^2=1024$ times, once for each combination of x_2,k_2. From the reconstructed 4D Wigner function we obtained the correlation function $\Gamma(x_1,x_2,0,x_1',x_2',0)$ by performing the 2D inverse Fourier transform in Eq. (3) 1024 times. Finally,

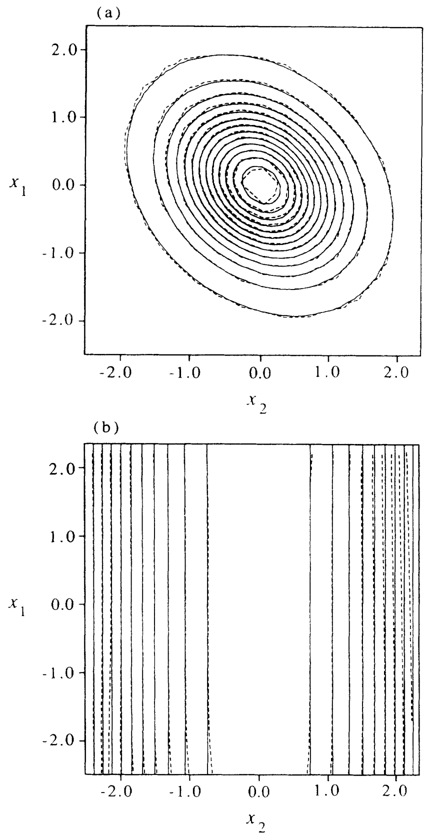

FIG. 2. Equal-separation contour plots of wave field for $\delta=1.3$, $\alpha=-0.25$. (a) Amplitude $|\psi(x_1,x_2,0)|$; (b) phase $\arg[\psi(x_1,x_2,0)]$. Solid curves are Eq. (12). Dashed curves are reconstructed using phase-space tomography. In (a) the lowest contour value is 3% of the amplitude's peak.

the wave field can be obtained by factorizing the correlation function in Eq. (1) because only a single wave function contributes to the average for the case of a coherent field. The reconstructed field is shown in Fig. 2, and is seen to be in good agreement with the original theoretical field. The present example is for a coherent field, for simplicity, but we have shown above that the method works also for partially coherent fields, where it returns the correlation function.

In conclusion, phase-space tomography has been introduced for the determination of the amplitude and phase structure of wave fields with arbitrary states of coherence (within resolution limits set by the number of angles and bins). Because it uses only refraction, rather than interference or diffractive optics, it represents a new class of phase retrieval methods for 2D fields. And because it requires no numerical iteration or deconvolution techniques, it solves the uniqueness and stability problems associated with other methods. The method might be especially useful in atom optics or photon-limited quantum optics because it obviates the need for complicated and/or lossy beam splitters and/or diffraction devices.

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