# Anomalous Scaling of a Randomly Advected Passive Scalar

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Solenoidal advection in $d$ dimensions may induce inertial-range behavior $\langle|T(\mathbf{x},t)-T(\mathbf{x}+\mathbf{r},t)|^n\rangle \propto r^{\zeta_n(T)}$ of a passive scalar field $T(\mathbf{x},t)$. If the velocity field changes very rapidly in time, the $\zeta_n(T)$ are determined by the two-particle eddy-diffusion coefficient $\eta(r) \propto r^{\zeta(\eta)}$. An approximation to the molecular-diffusion terms yields $\zeta_n(T) = \frac{1}{2}\sqrt{2nd[2-\zeta(\eta)]+\alpha^2} - \frac{1}{2}\alpha$ where $\alpha = d + \zeta(\eta) - 2$. As $n \to \infty$, $\zeta_n(T) \propto \sqrt{nd}$.

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Much attention has been paid to the inertial range of turbulence, which comprises wave numbers larger than those that contain most of the velocity variance and smaller than those directly affected by molecular dissipation [1]. The inertial range of an advected passive scalar field [2,3] promises to be more tractable theoretically. Both theory of the scalar inertial range and experimental data are reviewed by Sreenivasan [4]. A question recently addressed is the fractal dimension of isoscalar surfaces in the inertial range [5–10].

It is essential to recognize that advection does not fully determine the scalar structure functions in the inertial range of spatial separations. Molecular diffusion plays an important direct role. Only for the second-order structure function are diffusion effects easily isolated.

An exact expression can be written for advection effects on scalar structure functions if the velocity field changes very rapidly in time. In the present Letter, an approximation for the molecular-diffusion terms is used with this expression to obtain explicit predictions for the scaling of structure functions of all orders and for the dimension of isoscalar surfaces. It is believed that the results should remain valid for more general velocity fields.

The physics of the scalar-advection equation remains interesting if the Navier-Stokes (NS) velocity field is replaced by a stochastic field with prescribed power-law spectrum and higher statistics. There are exact results in the limit where the advecting field changes very rapidly in time: independent, closed differential equations can be constructed for the moments of each order. The equation for second-order moments (spectrum) has been solved in the inertial range [11]. The scaling of higher moments expresses any inertial-range intermittency growth.

Let the passive scalar field $T(\mathbf{x},t)$ obey

$$\left(\frac{\partial}{\partial t} + \mathbf{u}(\mathbf{x},t)\cdot\boldsymbol{\nabla}\right) T(\mathbf{x},t) = \kappa\nabla^2 T(\mathbf{x},t), \qquad (1)$$

where $\mathbf{u}(\mathbf{x},t)$ is a solenoidal velocity field and $\kappa$ is molecular diffusivity. If $\kappa = 0$, the level sets of $T(\mathbf{x},t)$ are material surfaces that move with the fluid.

Let $\Delta(\mathbf{x},\mathbf{x}',t) = T(\mathbf{x},t) - T(\mathbf{x}',t)$. Then (1) gives

$$\left(\frac{\partial}{\partial t} + \mathcal{A}(\mathbf{x},\mathbf{x}',t)\right) [\Delta(\mathbf{x},\mathbf{x}',t)]^n = \kappa K_n(\mathbf{x},\mathbf{x}',t), \qquad (2)$$

where

$$\mathcal{A}(\mathbf{x},\mathbf{x}',t) = \mathbf{u}(\mathbf{x},t)\cdot\boldsymbol{\nabla} + \mathbf{u}(\mathbf{x}',t)\cdot\boldsymbol{\nabla}',$$

$$K_n(\mathbf{x},\mathbf{x}',t) = n[\Delta(\mathbf{x},\mathbf{x}',t)]^{n-1}(\nabla^2 + \nabla'^2)\Delta(\mathbf{x},\mathbf{x}',t),$$

and $n$ is a positive integer. The left-hand side of (2) describes the evolution of $\Delta(\mathbf{x},\mathbf{x}',t)$ by simultaneous translation of the scalar amplitudes at $\mathbf{x}$ and $\mathbf{x}'$. The right-hand side smears small scales.

Suppose that the initial field $T(\mathbf{x},0)$ and any source term added to (1) have compact wave number spectra and vanishing odd moments and that there is an advection-mediated inertial cascade range of wave numbers $k = 2\pi/r$, larger than the initial or input wave numbers of $T$ and smaller than those at which molecular diffusion acts, such that the scalar structure functions obey

$$S_q(r) \equiv \langle|\Delta(\mathbf{x},\mathbf{x}+\mathbf{r},t)|^q\rangle \propto r^{\zeta_q(T)}. \qquad (3)$$

Here $q > 0$, $0 < \zeta_q(T) \leq q$, and $0 < \zeta_2(T) < 2$ (a condition discussed later). The largest scales of $T$ give an $O(r)$ contribution to $|\Delta(\mathbf{x},\mathbf{x}+\mathbf{r})|$ at small $r$. Therefore $\zeta_q(T) \leq q$ if (3) reflects advective cascade.

The left-hand side of (2) is linear in the field $[\Delta(\mathbf{x},\mathbf{x}',t)]^n$ and has a form independent of $n$. This is a direct expression of the fact that (1) has the infinite set of constants of motion $\int T^q d\mathbf{x}$ at $\kappa = 0$ [12]. The $n$ independence suggests that all the zero-mean fields $[\Delta(\mathbf{x},\mathbf{x}',t)]^m$ ($m$ odd) should suffer similar cascade and have the same inertial-range scaling: $\zeta_{pm}(T) = \zeta_p(T)$ $(p > 0)$. Hölder inequalities then imply

$$\zeta_q(T) = \zeta_1(T) \quad (q > 0). \qquad (4)$$

A simple model that satisfies (4) can be constructed as follows. Start with the field $T^G(\mathbf{x}) = \sum_{n=0}^{\infty} T_n(\mathbf{x})$ where the $T_n(\mathbf{x})$ are statistically independent Gaussian fields with spectra $\propto \exp(-\frac{1}{2}k^2/k_n^2)$ and equal variances. Here $k_n = 2^n k_0$ and $k_0$ is a macroscale wave number. Now smoothly modulate each $T_n(\mathbf{x})$ on a spatial scale $\gg 1/k_n$, using Gaussian envelopes, so that it is excited only in a fraction $\propto (k_n/k_0)^{-\mu}$ of the total volume, where $\mu$ is a number chosen in the range $0 < \mu < 1$. It is easy to verify that the resulting field $T(\mathbf{x})$ has the scaling (4),


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with $\zeta_1(T) = \mu$. The leading contribution to $S_2(r)$ comes from the band with $k_n \approx 2\pi/r$.

The dynamical basis of this model is the successive stretchings of parts of a scalar blob into ever thinner ribbons. The characteristic stretching time decreases with scale size, and the population density of ribbons of given thickness needed to maintain a statistically steady state is proportional to this time. The result is the scaled intermittency (4).

The $\kappa$ terms in (2) force scaling different from (4). For $q = 2$, it is well known [4,11] that an inertial range of wave numbers $k$ can exist in which velocity structure functions scale, $S_q(2\pi/k)$ scales, and direct effects of $\kappa$ are negligible. However, inertial-range $\kappa$ effects need not be negligible for $q \neq 2$. Suppose that $k_d = 2\pi/r_d$ is a characteristic wave number of the scalar dissipation range, so that molecular dissipation of Fourier amplitudes is negligible for $k \ll k_d$. For $k \ll k_d$, there are contributions to $\tilde{S}_4(k)$, the Fourier transform of $S_4(r)$, that are quadratic both in amplitudes with wave number $\approx k$ and in amplitudes with wave number $\approx k_d$. The latter amplitudes are strongly affected by $\kappa$, and the consequence is that $\tilde{S}_4(k)$ feels direct molecular dissipation.

By (2), the equation of motion for $S_{2n}(r)$ is

$$\frac{\partial S_{2n}(r)}{\partial t} + 2\langle \Delta^n \mathcal{A} \Delta^n \rangle = \kappa J_{2n}(r), \qquad (5)$$

where $r = |\mathbf{x} - \mathbf{x}'|$ and $J_{2n} = 2\langle \Delta^n K_n \rangle$. This is an analog of the Kolmogorov "4/5 law" [1]. Analytical treatment of inertial-range dissipation effects faces the difficulty that $J_{2n}(r)$ depends on four-point averages; in general, it cannot be expressed solely in terms of the $S_q(r)$ without approximation. But if $T$ has multivariate Gaussian statistics, or for any $T$ statistics at $n = 1$, $J_{2n}$ is exactly [13]

$$[J_{2n}(r)]_G = 2n(2n-1)S_{2n-2}(r)A(r), \qquad (6)$$

where $A(r) = \nabla^2 S_2(r) - \nabla^2 S_2(0)$.

A factor $S_{2n}/[(2n-1)S_{2n-2}S_2]$ can be introduced in the right-hand side of (6) to approximate inertial-range intermittency effects. Then,

$$J_{2n}(r) = 2nS_{2n}(r)A(r)/S_2(r). \qquad (7)$$

If $T$ is Gaussian or $n = 1$, (6) and (7) are identical. Under (4), the inertial-range intermittency factor scales as $r^{-\zeta_1(T)}$, and it can be verified explicitly that (7) gives a precise representation of $J_{2n}(r)$ in the model decribed following (4). Moreover, (7) is correct for an even simpler model of intermittency in which $T(\mathbf{x}, t)$ consists of sawtooth waves.

Equation (7) also describes a broader class of fractal models in which successively smaller scales of $T$ are increasingly intermittent and intensity of dissipation is spatially correlated with intensity of inertial-range excitation. A subclass of such models can be constructed from the model following (4) by making the amplitude distributions within the active regions scale with $k_n$ and correlating the positions of active regions for different $n$. For inertial-range $r$, the factor $A(r)/S_2(r) \approx -\nabla^2 S_2(0)/S_2(r)$ in (7) effectively replaces one pair of inertial-range amplitudes in $S_{2n}(r)$ by a pair of dissipation-range amplitudes, while leaving intact the strength of intermittency expressed by $S_{2n}(r)$.

In what follows, (7) is adopted as a plausible expression that can approximate $J_{2n}(r)$ with reasonable accuracy at inertial-range $r$ in the presence of a variety of types of intermittency and scaling. A modification of (7) that gives improved fidelity in the dissipation range is discussed after (14).

The exact $T$ spectrum induced under (1) by the inertial range of a NS velocity field is unknown. Precise results do exist if $\mathbf{u}(\mathbf{x}, t)$ has an infinitely short correlation time. In this limit, lowest-order perturbation analysis is exact [11]. It yields a closed partial differential equation for the evolution of each $n$th-order, $n$-point, one-time moment of $T(\mathbf{x}, t)$. Isotropic velocity-field statistics enter these equations only through the two-particle eddy diffusivity

$$\eta(r) = \frac{1}{2}\int_{-\infty}^{t} \langle [\delta_{\|} u(\mathbf{r}, t)\delta_{\|} u(\mathbf{r}, t')\rangle \, dt', \qquad (8)$$

where $\delta_{\|} u(\mathbf{r}, t) = [\mathbf{u}(\mathbf{x}, t) - \mathbf{u}(\mathbf{x} + \mathbf{r}, t)]\cdot\mathbf{r}/r$.

Let $L_0$ be a macroscale and assume a scaling range $k = 2\pi/r$ ($r \ll L_0$) where $\eta(r)$ has the form

$$\eta(r) = \eta_0 (r/L_0)^{\zeta(\eta)} \qquad (9)$$

and scalar dissipation is negligible. The differential equation for second-order moments yields [11]

$$\zeta_2(T) = 2 - \zeta(\eta) \quad [0 < \zeta(\eta) < 2]. \qquad (10)$$

Equation (10) is valid if interactions local in wave number dominate the cascade of scalar variance. The limit $\zeta(\eta) = 2$ corresponds to a velocity field confined to low wave numbers and a $k^{-1}$ scalar spectrum. At the limit $\zeta(\eta) = 0$, the dominant eddy-diffusivity contribution seen by a scalar wave number $k$ comes from much larger velocity-field wave numbers and acts at $k$ like an addition to $\kappa$; the scalar spectrum falls off exponentially. Both limits are discussed further below.

With the aid of homogeneity and incompressibility, $\langle \Delta^n \mathcal{A} \Delta^n \rangle$ can be evaluated exactly in the rapid-change limit to give

$$\langle \Delta^n \mathcal{A} \Delta^n \rangle = -\frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\eta(r)\frac{\partial S_{2n}(r)}{\partial r}\right), \qquad (11)$$

which contains the relative-diffusion operator introduced by Richardson [14]. The $n$-independent form of (11) reflects that of (2).

A white-in-time, statistically steady source added to (1) induces the term $2n(2n-1)S_{2n-2}(r)B(r)$ on the right-hand side of (5), where $B(r)$ is the product of second-order source structure function and source correlation time. Consider the steady state, with vanishing time derivatives, defined by (5) (including such a source), (7), and (11). The result for $S_2(r)$ in the inertial scaling range is

$$S_2(r) = \frac{\kappa\nabla^2 S_2(0)}{(\eta_0/L_0^2)\zeta_2(T)d}(r/L_0)^{\zeta_2(T)}, \qquad (12)$$

with $\zeta_2(T)$ given by (10). The numerator $\kappa\nabla^2 S_2(0)$ is the rate of dissipation of scalar variance. Note that $A(r) \approx -\nabla^2 S_2(0)$ for $r \gg r_d$.

Why do $\kappa$ terms make an explicit contribution to the equation for $S_2(r)$ in the inertial range? Note that $\langle[\Delta(\mathbf{x},\mathbf{x}',t)]^2\rangle = 2\langle T^2\rangle - 2\langle TT'\rangle$. Molecular diffusivity eats rapidly at the tiny dissipation-range contribution to $\langle T^2\rangle$ but not at $\langle TT'\rangle$, which has no significant dissipation-range contribution if $|\mathbf{x}-\mathbf{x}'|$ is an inertial-range separation.

Now consider $n > 1$. A crucial fact is that both (7) and (11) are linear in $S_{2n}(r)$ and, by (10), both go as $r^{\zeta_{2n}(T)-\zeta_2(T)}$. The forcing term is negligible in the inertial range. Thus the amplitude of $S_{2n}(r)$ cannot be determined by purely inertial-range analysis. The fit to the macroscale range must be examined for that. But $\zeta_{2n}(T)$ is fixed by the balance in (5) of the coefficients of $r^{\zeta_{2n}(T)-\zeta_2(T)}$ in (11) and (7). The balance equation is

$$\zeta_{2n}(\zeta_{2n} + d - \zeta_2) = nd\zeta_2, \qquad (13)$$

where (10) and (12) are used. The physical solution is

$$\zeta_{2n}(T) = \tfrac{1}{2}\sqrt{4nd\zeta_2 + (d-\zeta_2)^2} - \tfrac{1}{2}(d-\zeta_2). \qquad (14)$$

The asymptotic behavior of (14) is $\zeta_{2n}(T) \approx \sqrt{nd\zeta_2}$ $(n \to \infty)$. This is a growth of intermittency midway between the $\zeta_n(T) \propto n^0$ behavior (4) and Gaussian behavior $\zeta_n(T) \propto n$. All the Hölder realizability inequalities of the form $\zeta_q(T)/q \geq \zeta_p(T)/p$ $(q \leq p)$ are satisfied by (14) with real $q > 0$ replacing $2n$: the slope of $\zeta_q(T)$ as a function of $q$ decreases monotonically as $q$ increases.

Equation (14) survives if $\nabla^2$ in (1) is replaced by a hyperdiffusivity operator $-(-\nabla^2)^m$.

Equation (7) can be elaborated to express dissipation-range intermittency better by replacing $A(r)$ with $A(c_n r)$, where $c_n$ is a scaling factor that accounts for the dependence on moment order of the effective maximum-dissipation wave number. In the steady state with very long inertial range, $c_n$ can be determined self-consistently by taking $S_{2n}(r) \propto r^{2n}$ $(r \to 0)$ and forming a balance equation like (13).

Majda [15] uses rigorous renormalization analysis to obtain (10) for a class of $d = 2$ advection models under the assumption of rapidly changing velocity field. He deduces that the scalar statistics cannot be Gaussian if $0 < \zeta(\eta) < 2$.

The geometry of the surfaces on which $T(\mathbf{x},t)$ is constant is related to local statistics of $T(\mathbf{x},t)$ by the co-area formula of geometric measure theory [5]. In the inertial scaling range this yields the estimate

$$D_A(T) = d - \zeta_1(T) \qquad (15)$$

for the scaling dimension of the isoscalar surface within a sphere whose radius is an inertial-range interval. If (14) is continued to $n = 1/2$ it gives

$$\zeta_1(T) = \tfrac{1}{2}\sqrt{2d\zeta_2 + (d-\zeta_2)^2} - \tfrac{1}{2}(d-\zeta_2). \qquad (16)$$

The limit case $\zeta(\eta) \to 2$ is realized at $r$ much smaller than the reciprocal of the largest wave number significantly represented in the velocity field. The velocity differences in (10) then are $\propto r$ while the time scale is the lesser of the correlation time and the reciprocal of the typical shear, independent of $r$. In this range, (12) gives $\zeta_2(T) \to 0$, corresponding to a $k^{-1}$ scalar-field spectrum. The physics is well studied [16,17]. Scalar fluctuations are strained in a self-similar fashion, so that there is no increase of intermittency with $1/r$ within the range, and all $\zeta_n(T) \to 0$. Equation (15) gives $D_A(T) \to d$: the level sets are space filling [9,18].

In the limit $\zeta(\eta) = 0$, the effective eddy diffusivity acting on a scalar wave number $k$ is a logarithmically divergent integral over velocity wave numbers. The eddy diffusivity then acts like a renormalization of $\kappa$. As $\zeta(\eta) \to 0$, $\zeta_2(T) \to 2$ and the scalar inertial-range spectrum approaches $k^{-3}$. At the same time, the effective diffusivity grows and drives the coefficient of the spectrum toward zero while the range of $k$ required to nearly realize the asymptotic spectrum form increases. At the limit, the level of the power-law spectrum is zero, and it is replaced by exponential decay.

Without appeal to (14), the realizability constraint $2\zeta_1(T) \geq \zeta_2(T)$, used with (10) and (15), gives $D_A(T) \leq d-1+\zeta(\eta)/2$. Thus, realizability requires $D_A(T) \to d-1$ as $\zeta(\eta) \to 0$ and the scalar spectrum exponent approaches $-3$. If, as under (14), there is intermittency of $\Delta(\mathbf{x},\mathbf{x}+\mathbf{r},t)$ that grows with $1/r$, then $2\zeta_1(T) > \zeta_2(T)$. This implies $D_A(T) = d-1$ already for some $\zeta(\eta) > 0$. $D_A(T)$ sticks at $d-1$ from this value of $\zeta(\eta)$ to $\zeta(\eta) = 0$. As noted after (3), $\zeta_1(T) > 1$ is impossible because of macroscale contributions.

If the velocity field does not change very rapidly in time, but remains Gaussian, $\eta(r)$ plausibly is still the primary determinant of scalar behavior, and (14) may be expected to remain approximately correct. Lagrangian velocities should be used in (10) when the Eulerian velocity field changes slowly [11]. A Gaussian velocity field that changes slowly in time, and has a long scaling range $0 < \zeta_2(u) < 2$, acts like a rapidly changing field because the large scales sweep fluid elements rapidly through the small scales. Here $\zeta_n(u)$ is the scaling exponent of $\langle|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}+\mathbf{r})|^n\rangle$. The sweeping implies $\zeta(\eta) = \zeta_2(u) + 1$.

Scale-dependent intermittency of the velocity field may be expected to induce additional intermittency in the advected scalar field, thereby decreasing the growth of $\zeta_{2n}(T)$ with $n$.

Constantin and Procaccia have presented an estimate for $D_A(T)$ equivalent via (15) to

$$\zeta_1(T) = \tfrac{1}{2} - \tfrac{1}{2}\zeta_1(u). \tag{17}$$

The analysis is based on geometric measure theory and associated bounds on the fractal dimension of the graph of the function $T(\mathbf{x}, t)$ in $(d+1)$ space [5,7,8]. Equation (17) conflicts with (10) plus (16), but the quantitative difference is small if $\zeta(\eta) = 4/3$, $\zeta_1(u) = 1/3$ (classical values [2,3,14]). In the case of rapidly changing velocity field, it is exact that $T$ scaling depends only on $\zeta(\eta)$, which does not have a one-to-one relation to $\zeta_1(u)$.

Antonia *et al.* [19] report measurements of $\zeta_n(T)$ in jets and atmospheric boundary layers that vary from 0.65 at $n = 2$ to 1.82 at $n = 12$. For comparison, if $d = 3$ and $\zeta_2(T) = 2/3$, (14) gives $\zeta_1(T) \approx 0.37$, $\zeta_{12}(T) \approx 2.49$. Meneveau *et al.* [20] report other jet determinations of $\zeta_n(T)$, via multifractal analysis, that rise with $n$ at low $n$ but bend over to nearly $n$-independent values at larger $n$. In the jet studies, the scaling ranges of the data are short, there is inhomogeneity, the velocity fields are intermittent and decaying, and the source of the scalar fluctuations is the turbulent mixing of a sharp interface. This driving mechanism inputs fluctuations at all scales, in contrast to the homogeneous low-$k$ excitation assumed here. It is hard to justify a quantitative comparison with the present predictions. The approach to asymptotic statistics as a scalar inertial range grows in extent can be very slow [4].

Existing massively parallel supercomputers permit robust $d = 2$ simulations of (1) with a prescribed stochastic velocity field at spatial resolutions of $8192^2$ or even higher [21]. Initial-value problems can be solved using (5), (7), and (11), and the resulting predictions for $S_{2n}(r)$ can be tested quantitatively, at finite Peclét numbers, against simulations that have the same initial conditions. This may reduce the urge to guess what asymptotics are hinted at by the simulations.

Equations (13) and (14) change if the approximation (7) for the dissipation terms is replaced by something else. The sensitivity to change is reduced somewhat by the fact that $\zeta_{2n}$ appears quadratically in (13). If $J_{2n}(r)$ is to be approximated in terms of the $S_q(r)$, the choice is restricted by the requirement that $J_{2n}(r)$ be represented exactly in the limits examined after (7), the Gaussian case and the particular intermittent models that are discussed explicitly. Thus, one cannot simply insert a function of $n$ in front of (7). Whatever the degree of verity of (7) and (13), they serve as an example of how the dissipation terms can affect the scaling of higher structure functions in the inertial range.

If $\kappa = 0$, (5) and (11) give the exact evolution of $S_{2n}(r)$ for the case of rapidly changing velocity field. The scalar level sets are then material surfaces. This is a very treacherous regime to interpret. A growing scalar range with $k^{-1}$ spectrum appears at $k$ higher than the velocity-field cutoff wave number. This range has nontrivial effects on $S_{2n}(r)$ that increase with time and are hard to remove consistently.

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