Superradiant Laser

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We discuss a novel laser scheme in which N three-level atoms maintain full cooperativity in a stationary regime. The intensity I of radiation is proportional to N^2, the linewidth scales as $1/N^2$, and the output intensity fluctuations display up to 100% squeezing at low frequencies. The possibility of simultaneously slowing down phase diffusion and reducing intensity fluctuations makes this kind of laser an attractive goal to go for.

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When several identical two-level atoms are brought to their excited state with no external electromagnetic field imposed then or later, the subsequent radiative deexcitation can, certain conditions met, proceed collectively, i.e., with all N atoms acting like one rigid dipole [1,2]. The ensuing superfluorescent radiation pulse has a peak intensity $I_s \sim N^2$ while noncollective radiation would only yield $I_n \sim N$. Since the maximum energy available for radiation is N times the single-atom excitation energy $\hbar\omega$, the temporal width of a superfluorescent pulse must be inversely proportional to N ($\tau_s \sim 1/N_\gamma$) while the radiative lifetime $1/\gamma$ of the excited state of a single atom would equal the duration τ_n of a pulse of normal fluorescence. For experimental realizations of superfluorescence [3] one must, roughly speaking, make sure that the characteristic times of all competing processes like inhomogeneous broadening and collisions are much longer than τ_s. We shall here be concerned with a new and rather different type of superradiance. Like the aforementioned one, it is collectively generated by N atoms and thus has an intensity $\sim N^2$; unlike the former, it can be stationary rather than transient. An even more striking difference arises for the spectral width: While a superfluorescent pulse has a spectral width $\sim \gamma N$, the linewidth of the superradiant laser to be discussed presently can be extremely small, $\Delta\nu \sim 1/N^2$. Moreover, the intensity fluctuations within an individual superfluorescent pulse are close to those of a coherent state; those of the stationary output of a superradiant laser can be much smaller and in fact can be squeezed nearly perfectly. We shall also discuss, at the end of this Letter, the degradation of superradiance resulting when full cooperativity of all atoms is broken; our scheme then reduces to a more conventional Raman laser with $I \sim N$, $\Delta\nu$ independent of N, and no more than 50% squeezing of the intensity fluctuations.

The simplest model of a superradiant laser accounts for three-level atoms (see Fig. 1) placed inside a resonator. A classical monochromatic wave in resonance with the transition $0 \longleftrightarrow 2$ serves as a pump. The lasing cavity mode is taken in tune with the transition $2 \longleftrightarrow 1$. Finally, a certain collective relaxation $1 \rightarrow 0$ recycles the atoms back to the influence of the pump. We now turn to a more detailed specification of the three partial processes in play. The Hamiltonian

$$H = i\hbar g(aS_{21} - a^\dagger S_{12}) + i\hbar\Omega(S_{20} - S_{02}) \tag{1}$$

displays the collectivity of the pump mechanism and of the interaction of the atoms with the lasing mode by the appearance of the collective polarization operators $S_{ij} = \sum_{\mu=1}^{N} S_{ij}^{\mu} = \sum_\mu (|i\rangle\langle j|)^\mu$. There are nine operators S_{ij} —those with $i \neq j$ refer to polarizations while each "diagonal" one, $S_{ii} \equiv P_i$, measures the global occupation of level i—they obey $S_{ij}^\dagger = S_{ji}$ and $[S_{ij}, S_{kl}] = \delta_{jk}S_{il} - \delta_{il}S_{kj}$.

The operators a and $a^\dagger$ annihilate and create photons of the lasing mode. The amplitude Ω of the external classical pump field and the coupling constant g are specified in the Hamiltonian (1) so as to have the dimension of a frequency. In fact, Ω is the frequency of the Rabi oscillations which the pump tends to impose on the transition $0 \longleftrightarrow 2$: The coupling constant g has an analogous meaning for the transition $1 \longleftrightarrow 2$ but is referred to the fictitious field of a single photon. Note that by taking Ω as a fixed c number we formally forbid pump fluctuations.

Two damping mechanisms must be accounted for. The irreversible leakage of laser photons through a nonideal mirror is commonly modeled so as to add to the time rate of change of the mode amplitude a the term [4–6]

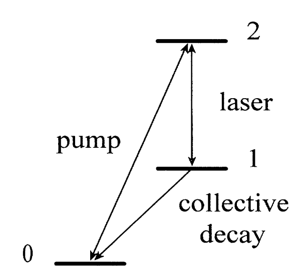

FIG. 1. Three-level scheme and transitions.

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$$\left(\frac{\partial a}{\partial t}\right)_{\text{irr}} = -\kappa a(t) + \sqrt{2\kappa}\,\xi(t) , \tag{2}$$

with the damping constant κ and the quantum Langevin force $\xi(t)$. The latter force ensures the preservation of the Bose commutator $[a(t),a^\dagger(t)]=1$ at all times: For our purpose we can take it to have Gaussian statistics and a white spectrum according to

$$\begin{aligned} &[\xi(t),\xi^\dagger(t')] = \langle \xi(t)\xi^\dagger(t')\rangle = \delta(t-t') , \\ &\langle \xi(t)\rangle = \langle \xi^\dagger(t)\xi(t')\rangle = 0 . \end{aligned} \tag{3}$$

We may interpret the operators ξ and $\xi^\dagger$ as representatives of the vacuum fluctuations of the electromagnetic field outside the resonator.

The atomic relaxation from level 1 back to the ground state 0 must be a fully collective one if the laser is to superradiate. Such cooperative damping will occur if the transition in question is coupled to another mode of the resonator. However, with respect to the latter "passive" mode the resonator need not have a high finesse. We shall in fact assume the simplest situation serving our purpose, i.e., the one allowing immediate adiabatic elimination of the passive mode. That situation would, in the absence of level 2 and with level 1 populated initially, entail the no-ringing limit of superfluorescence on the transition $1 \to 0$, as observed in [3]. The ensuing irreversible additions to the time rates of change of the atomic observables involve a damping constant γ and a noise force $\eta(t)$ as [4,5,7]

$$\begin{aligned} &\left(\frac{\partial S_{02}}{\partial t}\right)_{\text{irr}} = \gamma S_{12}S_{01} + \sqrt{2\gamma}\,S_{12}\eta , \\ &\left(\frac{\partial S_{12}}{\partial t}\right)_{\text{irr}} = -\gamma S_{10}S_{02} + \sqrt{2\gamma}\,\eta^\dagger S_{02} , \\ &\left(\frac{\partial S_{01}}{\partial t}\right)_{\text{irr}} = \gamma(P_1-P_0)S_{01} + \sqrt{2\gamma}(P_1-P_0)\eta , \\ &\left(\frac{\partial P_0}{\partial t}\right)_{\text{irr}} = -\left(\frac{\partial P_1}{\partial t}\right)_{\text{irr}} \\ &\qquad = 2\gamma S_{10}S_{01} + \sqrt{2\gamma}(S_{10}\eta + \eta^\dagger S_{01}) . \end{aligned} \tag{4}$$

The stochastic force $\eta(t)$ plays the same role for the "passive" mode as $\xi(t)$ for the "active" one—Eqs. (3) thus hold for $\eta(t)$ as well—moreover, $\eta(t)$ is independent of $\xi(t)$. Two features of the collective relaxation terms (4) are worth a comment, the nonlinearity of the damping and the "multiplicative" form of the noise: Both of these features have as their common origin the nonlinearity of the interaction of the atoms with the passive mode. They entail the conservation of the operator [8]

$$\sum_{i,j} S_{ij}S_{ji} = N(N+2) , \tag{5}$$

which generalizes the better known conservation of the squared length of the Bloch vector in the collective dynamics of two-level atoms. A second conservation law, $\sum_i P_i = N$, is respected both by the Hamiltonian (1) and the damping (2)-(4); clearly, none of the processes accounted for is capable of changing the number of atoms.

Admittedly, the laser model just presented is a bit of an oversimplification. Pump fluctuations, detunings, competition of spontaneous emission, further atomic levels, or the replacement of one of the three transitions $i \leftrightarrow j$ by a two-photon transition will eventually deserve discussion. By leaving such refinements aside for now we hope to exhibit most clearly but without inappropriate exaggeration the potential of cooperativity for noise reduction. At any rate, with its five parameters N, g, Ω, κ, and γ even our simple model already presents a wealth of modes of behavior. Quite amazingly, though, the Heisenberg equations of motion for the eleven observables $S_{ij}, a, a^\dagger$,

$$\dot{X} = (i/\hbar)[H,X] + (\partial X/\partial t)_{\text{irr}} , \tag{6}$$

allow for analytic treatment, at least in certain interesting limits.

In the following we confine ourselves to the semiclassical limit $N \gg 1$. Each of the eleven observables can then be represented as a sum $X = \bar{X} + \delta X$ of a dominant classical term $\bar{X} \sim N$ and a "small" operator valued fluctuation. Of course, the proportionality of the means $\bar{X}$ to N is a manifestation of the full cooperativity assumed. To find the $\bar{X}$ in the stationary regime we drop $\dot{X}$ and the noise forces in the Heisenberg equations and degrade each operator X to a c number $\bar{X}$. The resulting solutions can be expressed in terms of two combinations of the five parameters in play. One of them is a normalized dimensionless coupling strength c, the other an effective pump strength p:

$$c = g^2/\kappa\gamma, \quad p = |\Omega|/N\gamma\sqrt{c} . \tag{7}$$

Formally, the classical equations allow for two different stationary solutions. In order to be physically acceptable, these solutions must obey two restrictions which have a quantum mechanical origin and are not built into the classical equations of motion. The restrictions in question are (α) $0 \le \bar{S}_{ii}/N \le 1$ (due to $\bar{S}_{ii}/N$ being probabilities) and (β) Schwartz's inequality $\bar{S}_{ii}\bar{S}_{jj} \ge |\bar{S}_{ij}|^2$ which must hold for all pairs ij of levels. It is a most interesting and surprising manifestation of quantum effects on macroscopic scales that the restrictions (α,β) rule out one of the formally arising stationary solutions. The acceptable one has the field amplitude

$$\bar{a} = \frac{N\gamma}{g}\frac{c}{\sqrt{1+c}}\sqrt{p(1-p)} , \tag{8}$$

the occupation probabilities $\bar{S}_{22}/N = (1-p)/(1+c)$, $\bar{S}_{11}/N = p$, and the polarizations $|\bar{S}_{ij}|^2 = \bar{S}_{ii}\bar{S}_{jj}$, the latter equalities reflecting full cooperativity. Obviously, the pump parameter must be restricted to $0 \le p \le 1$. We should note that all stationary means $\bar{X}$ can be taken as real, as is exemplified for the field amplitude in (8). In fact, such a choice fixes a phase not determined by the classical equations of motion; quantum fluctuations, to be accounted for presently by the operators δX, cause that phase to undergo diffusion (see below). The absence of a threshold for the pump strength p in the mode amplitude (8) can be seen as due to our neglect of spontaneous emission. A less expected feature of the stationary mode amplitude is the appearance of an upper limit for the pump strength, $p=1$, and of an optimal pumping, $p=\frac{1}{2}$, at which $\bar{a}$ is maximal. As a final remark on the mode amplitude $\bar{a}$ in (8) we would like to once more underscore the proportionality $\bar{a}\sim N$ which manifests the superradiant character of the laser in discussion.

To check on the stability of the stationary solution and to find the fluctuations $\delta X(t)$ one may linearize the Heisenberg equations with respect to the fluctuations δX. That linearization also brings about a rather beneficial simplification of the noise: Through $S_{ij}(t)\eta(t)\rightarrow \bar{S}_{ij}\eta(t)$ the atomic noise forces in (4) are freed of their so-called multiplicative character, i.e., they become simple inhomogeneous terms. Splitting the fluctuations into Hermitian "real" and "imaginary" parts as $\delta a=\delta u+i\delta v$ and, for $i\neq j$, $\delta S_{ij}=\delta u_{ij}+i\delta v_{ij}$, we arrive at linear inhomogeneous equations for the eleven fluctuation operators δu, δv, δu_{ij}, δv_{ij}, and $\delta S_{ii}=\delta P_i$. Because of the reality of $\bar{a}$ and $\bar{S}_{ij}$ the linearized equations fall into two separate blocks, one for the four imaginary parts δv, δv_{ij} $(i\neq j)$, and the other for the seven real parts δu, δu_{ij}, and δP_i; obviously, the imaginary parts may be interpreted as phase fluctuations through $\delta v=\bar{a}\delta\varphi$, etc.

Stability depends on the three parameters c, p, and $N\gamma/\kappa$. We shall restrict ourselves in the present paper to discussing the "adiabatic" limit $N\gamma/\kappa\rightarrow\infty$. In that case a necessary condition for stability is $p>(1-c)/2$ which for $c<1$ introduces a pumping threshold. A second stability condition restricts the pump strength from above, $p\leq c/(1+2c)$. These restrictions arise for the phase fluctuations. By contrast, the independent equations for the amplitude fluctuations are reduced in number from 5 to 1 in the adiabatic limit and have a single characteristic attenuation rate,

$$1/\tau=4\kappa(1+c)(1-p)/(3+c-2p)\,, \tag{9}$$

which is incapable of going negative.

We have solved for the frequency dependent phase fluctuations $\delta v(\omega)=\int_{-\infty}^{+\infty}dt\,e^{+i\omega t}\delta v(t)$. The linewidth $\Delta\nu$ of our superradiant laser is then accessible through $\langle\delta v(\omega)\delta v(\omega')\rangle=2\pi\delta(\omega+\omega')|\bar{a}|^{-2}\Delta\nu/\omega^2$. The following simple expression is obtained in the limit $N\gg1$, $N\gamma\gg\kappa$, $\omega\lesssim\kappa$,

$$\Delta\nu=\frac{\kappa^3}{g^2N^2}\frac{(1+c)[p^2(1+c)^2+(1-p)^2(1-c)^2]}{p(1-p)(-1+c+2p)^2}\,. \tag{10}$$

The dependence on the number of atoms, $\Delta\nu\approx1/N^2$, offers the interesting possibility of line narrowing by collective radiation. To fully exploit that potential one would have to strive for pump and coupling strengths of order unity [9]. Incidentally, phase diffusion is manifested by the low-frequency divergence of

$$\delta v(\omega),\langle\delta v(\omega)\delta v(\omega')\rangle\sim1/\omega^2\,.$$

In view of the phase diffusion just discussed, the real part δu of the mode amplitude a is distinguished, among all linear combinations of a and $a^\dagger$ with the form $ae^{-i\theta}+a^\dagger e^{i\theta}$, by the greatest potential for squeezing at low frequencies [10,11]. The amplitude correlation function $\langle\delta u(\omega)\delta u(\omega')\rangle$ is thus of special interest. An appealingly simple result is again obtained in the adiabatic limit $N\gamma/\kappa\gg1$. We quote the corresponding correlation function for the field $a_{\text{out}}(t)=\sqrt{2\kappa\tau_0}a(t)-\sqrt{\tau_0}\xi(t)$, where τ_0 is the cavity round trip time, transmitted to the outside of the cavity [5] through the outcoupling mirror,

$$\langle\delta u_{\text{out}}(\omega)\delta u_{\text{out}}(\omega')\rangle\sim\delta(\omega+\omega')\left[1-\frac{S(p,c)}{1+\omega^2\tau^2}\right], \tag{11}$$

where positive values of the "squeezing,"

$$S(p,c)=\frac{(1+6c+c^2)(1-2p)+4cp^2}{2(1+c)^2(1-p)^2}\,, \tag{12}$$

indicate noise reduction below the vacuum level. Obviously, then, the fluctuation spectrum in (11) displays a Lorentzian dip of the width $1/\tau$ given in (9), centered at zero frequency. Ideal squeezing, $S=1$, is implied by (12) at $c=1,p=0$. The plot in Fig. 2 reveals a flat maximum of $S(p,c)$ near the point of ideal squeezing.

As another indicator of nonclassical fluctuations, we have calculated Mandel's parameter $Q=\langle(n-\langle n\rangle)^2\rangle/\langle n\rangle-1$ where $n=a^\dagger a$ is the stationary photon number inside the cavity. This is immediately accessible from $a=\bar{a}+\delta a$ and the double Fourier transform of the correlation function $\langle\delta u(\omega)\delta u(\omega')\rangle$. The result is $Q(p,c)=-S(p,c)\times(1-c)(1-p)/(3+c-2p)$. Interestingly but not unex-

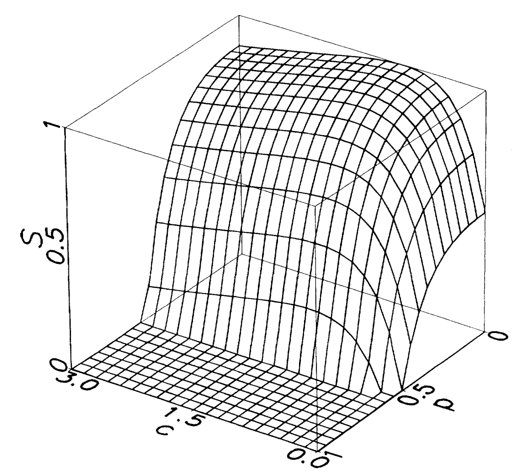

FIG. 2. The squeezing function $S(p,c)$ of the collective laser according to (12). $S=1$ corresponds to optimal squeezing, $S=0$ to a coherent state.

pectedly, negative Q and positive squeezing S are concomitant. The minimal value of Mandel's parameter, $Q(0,1+\sqrt{8}) \approx -0.59$, is a little lower than midway between the coherent-state value $Q=0$ and the absolute minimum $Q=1$ attained for an eigenstate of the photon number. Incidentally, the point of optimal squeezing $(p=0,c=1)$ is not the one of minimal Q; instead, $Q(0,1)=-\frac{1}{2}$.

It is good to compare the superradiant laser under discussion with a similar but noncollectively radiating one. To that end, we break the collectivity by letting the relaxation $1 \to 0$ take place independently in each atom. The irreversible rates of change (4) are then modified according to $S_{ij}S_{kl} \to \sum_{\mu=1}^{N} S_{ij}^{\mu} S_{kl}^{\mu} = \delta_{jk} S_{il}$ and $S_{ij}\eta \to \sum_{\mu} S_{ij}^{\mu} \eta_{\mu}$. The single-atom noise forces η_μ are all independent; each of them has properties like those given for ξ in (3). Our reference system thus is an ordinary Raman laser akin to that recently discussed by Ritsch, Marte, and Zoller [12]. Clearly, the stationary number of photons now must scale linearly rather than quadratically with N. We find, for $N \gg 1$,

$$|\bar{a}_n|^2 = \frac{N\gamma}{\kappa} \sqrt{p_n(1-p_n)/2}\,, \tag{13}$$

where the "noncollective" pump parameter $p_n = 2\Omega^2\kappa/N\gamma g^2$ must range in the interval [0,1] for the stationary solution to exist. A conventional noise analysis yields the line width, for $Nc \gg 1$,

$$(\Delta\nu)_n = \frac{\kappa g^2}{(\gamma+\kappa)^2} \left[\frac{p_n}{2(1-p_n)} \right]^{1/2}. \tag{14}$$

The asymptotic independence of $(\Delta\nu)_n$ of N is in striking contrast to the scaling $\Delta\nu \sim 1/N^2$ in the superradiant case. Finally, the output amplitude fluctuations come out as in (11) but with $1/\tau_n = 4\kappa(1-p_n)$ and the squeezing $S_n(p_n)$ depending on p_n as

$$S_n(p_n) = \frac{1}{4(1-p_n)^2} [2 - 4p_n^2 - (1+p_n)\sqrt{2p_n(1-p_n)}]\,. \tag{15}$$

It follows that the maximum squeezing, $S_n(0)=\frac{1}{2}$, is attained at the cusp of $S_n(p_n)$ at vanishing pumping. A flatter maximum resides at $p_n \approx 0.11$ but has only $S_n(0.11) \approx 0.46$. At any rate, the zero-frequency squeezing in the noncollective laser is rather inferior to the 100% noise reduction met with above for the superradiant case.

Mandel's Q now comes out as $Q_n(p_n) = -(1-p_n) \times S_n(p_n)$ which rises monotonically with increasing pump strength p_n from $Q_n(0) = -\frac{1}{2}$ to $Q_n(p_n \approx 0.48) = 0$, from which latter point on one encounters "super-Poissonian" amplitude noise. Again, we find noisier behavior in the noncollective case than for the superradiant laser.

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