Tip Dynamics in Saffman-Taylor Fracture

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(Received 4 February 1993)

We present a model for a Saffman-Taylor finger advancing into a viscoelastic medium, along with solutions for the dynamics of the tip of the finger coupled to a "fractured zone." The fractured zone is defined as a region of the medium which the finger can penetrate, and ahead of which stress builds up, eventually leading to fracture of the medium. The model predicts a self-sustained oscillation in the tip velocity arising via a Hopf bifurcation.

PACS numbers: 47.20.Ky, 46.30.Nz, 47.55.Kf, 83.50.Fc

An interesting discovery that has recently attracted considerable attention is the existence of fracture instability in viscous fingering [1-4], a process which we term "Saffman-Taylor fracture." Ordinary fingering occurs when a less viscous fluid, such as water, pushes a more viscous fluid, such as oil. Saffman-Taylor fracture can occur when the oil is mixed with very fine grains or powder. These grains make the oil viscoelastic, so that it can respond elastically to stress. If the flow of the less viscous invading fluid is sufficiently slow, ordinary viscous fingering still occurs. However, if the time scale τ_f for the flow is shorter than the response time τ_r of the medium, the process is dramatically different: The medium fractures, and the fingers become much narrower and split much less often [1]. In this regime the zero shear stress, or yield stress, in the viscoelastic medium is finite [5], unlike in ordinary fluids, for which it vanishes. Thus the finger front can only advance when it fractures the medium, i.e., when the fluid applies a stress to the medium which exceeds some threshold.

There are several other physical systems whose dynamics involve a threshold. Examples include the following: (i) The dynamics of a grain in a densely packed granular assembly, where each grain is surrounded by neighbors and hence is locked. The grain does not move unless an applied force exceeds a critical strength, after which large-scale motion suddenly sets in [6]. (ii) The peeling of an adhesive tape [7,8]. In the peeling process, we hear a distinctive sound with a characteristic frequency. Thus there exists a characteristic time, which is probably the time required to build up the threshold stress. (iii) Dynamic fracture in brittle materials. The need for the threshold in dynamic fracture of brittle materials has also been quite convincingly demonstrated in PMMA [8]: The displacement of a crack tip vs time exhibits steplike behavior, suggesting that in order for the crack to advance, a critical stress must be supplied. We recognize that an essential ingredient in the dynamics of such physical systems appears to be the existence of a threshold for initiating the dynamical process.

The purpose of this paper is twofold. First, we propose a dynamic model for Saffman-Taylor fracture by paying particular attention to the dynamics of the tip of the finger front coupled to a *fractured zone*, which will be defined shortly. Second, we analyze the model equations and show that they display one of the generic features of threshold-induced dynamics, namely, the appearance of a self-sustained oscillation in the front dynamics. This oscillation is similar to the stick-slip processes commonly observed in the systems enumerated above. Even though the viscous effect dominating in the experiments of Refs. [1] and [3] might not be crucial in dynamic fracture, our picture advanced in this paper might offer some new insight into the recent observation of dynamic oscillations in the tip velocity of a crack propagating into a brittle material [9-11].

Consider a finger of fluid being forced into a viscoelastic medium in the x direction (Fig. 1). Experimental pictures of Saffman-Taylor fracture in a Hele-Shaw cell [1-3] seem to indicate that the usual two-dimensional approximations do not hold, because the thickness of the finger is comparable to the cell thickness. Thus we regard the finger as a three-dimensional object. Far behind the tip, the finger has a cross-sectional area Σ_∞, and fluid is being forced into it at a speed V_0, for a volumetric flow rate of $Q \equiv \Sigma_\infty V_0$. We assume that the tip of the finger advances into a compliant, or "fractured," zone of the medium, which can be experimentally determined by measuring the stress field distribution near the finger tip. One may define a similar zone near the advancing front

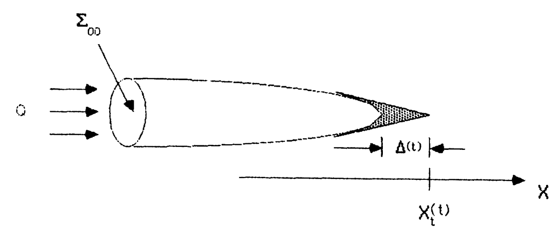

FIG. 1. A finger of cross-sectional area Σ_∞ advances into a viscoelastic medium. When the flow rate τ_f is much shorter than the response time of the medium τ_r, stress builds up ahead of the tip of a "fractured zone." The tip position of the fractured zone is denoted by x_t. The fluid flux at infinity is Q.

0031-9007/93/71(6)/847(4)$06.00

in the peeling of an adhesive tape. The rest of the medium is "rigid," and the fluid cannot penetrate it. The tip of the fractured zone is at $x=x_t(t)$, a distance $\Delta(t)$ ahead of the tip of the finger. As the finger advances toward the tip of the fractured zone, the pressure needed to keep pumping fluid into it at a rate Q increases, and stress builds up ahead of the fractured zone. Thus Δ simultaneously measures the pressure in the fluid, the curvature of the finger tip, and the accumulated stress in the medium: The same process which decreases Δ increases all three. Ultimately the stress becomes so large that the medium ahead of the zone fractures, thus advancing the zone.

We now derive the dynamic equations for the tip position of the fractured zone, $x_t(t)$, and that of the finger tip, or equivalently $\Delta(t)$. First note that increasing x_t by dx_t, while holding Δ fixed, corresponds to shifting the entire arrangement of finger and fractured zone forward by dx_t. This increases the volume of the finger by $\Sigma_\infty dx_t$. Holding x_t fixed and *decreasing* Δ by $d\Delta$ corresponds to advancing the finger into a fixed fractured zone. In order for this to happen, the pressure must increase and so the curvature of the tip must also increase. Thus the volume of the finger increases by *less* than $\Sigma_\infty d\Delta$. We will denote the increase by $\Sigma_\infty A(\Delta)d\Delta$; we may think of $\Sigma_\infty A(\Delta)$ as the effective cross-sectional area of the finger tip when it is a distance Δ behind the tip of the fractured zone. All the complications involved in solving the free-boundary problem for the shape of the finger are incorporated into the function $A(\Delta)$. If the fluid is incompressible, then the volume occupied by the finger must always increase at the rate Q. Thus we have

$$Q=\Sigma_\infty dx_t/dt-\Sigma_\infty A(\Delta)d\Delta/dt \tag{1}$$

or

$$V_0=V-A(\Delta)d\Delta/dt\,, \tag{2}$$

where $V_0=Q/\Sigma_\infty$, and we have defined $V(t)$ to be the rate of advance of the tip of the fractured zone,

$$V(t)\equiv dx_t/dt\,. \tag{3}$$

As the finger tip advances relative to the fractured zone, i.e., as Δ decreases, the stress ahead of the fractured zone increases. This stress governs the rate of advance of the tip of the fractured zone. Any reasonable constitutive relation for a viscoelastic medium must incorporate two regimes. For low stress levels, the medium can flow slowly as a viscous fluid, so that the tip of the fractured, or compliant, zone can advance slowly. For high stress levels, the medium cannot rearrange itself on a sufficiently fast time scale, and so it behaves as an elastic solid and fractures. In an intermediate stress range, however, both processes are possible: The stress is not sufficient to *initiate* fracture, but if fracture has already begun, then the stress is sufficient to *maintain* it. We might then expect that the speed V at which the tip of the fractured zone advances would be a function of the stress, or equivalently some function $f(\Delta)$, which must have at least two branches: a high-stress, high-speed branch $f_+(\Delta)$ and a low-stress, low-speed branch $f_-(\Delta)$. This behavior is sketched in Fig. 2. The upper branch ends at $\Delta=\Delta_+$, above which the stress is no longer sufficient to maintain fracture, and the lower branch ends at $\Delta=\Delta_-$, below which the stress is so high that fracture is initiated. (Both branches have a slight downward slope, since increasing Δ means decreasing stress, which should slow the advance of the fractured zone slightly.) Such a model would be inadequate in at least two respects. First, it offers no way of making the transition between the two dynamical regimes—as Δ increases beyond Δ_+ the velocity can only jump instantaneously from the upper branch to the lower. Since this would imply infinite acceleration of material, we must assume instead that the fractured zone adjusts to changes in the stress field on a fast, probably microscopic time scale τ. This leads generically to an equation of the form

$$\tau dV/dt+V=f(\Delta)\,. \tag{4}$$

The second difficulty with a model of this type is purely mathematical. If $f(\Delta)$ is multiple valued, then Eq. (4) is, strictly speaking, meaningless. We will attend to this problem after first investigating some simple consequences of Eqs. (2) and (4) and showing that it cannot simply be ignored.

If we ignore the difficulty that $f(\Delta)$ is multiple valued, then Eqs. (2) and (4) comprise a complete two-dimensional dynamical system specifying the evolution of the finger as described by the two variables V and Δ. Fixed points of this dynamical system correspond to states in which the tips of the finger and the fractured zone advance together at a constant speed. By setting the time derivatives of V and Δ to zero, we find that a fixed point occurs at $V=V_0$ and $\Delta=\Delta_0$, where $f(\Delta_0)=V_0$. To test the stability of the fixed point, we write $V(t)=V_0+\hat{V}e^{\sigma t}$ and $\Delta(t)=\Delta_0+\hat{\Delta}e^{\sigma t}$ in (2) and (4), and linearize in $\hat{V}$ and $\hat{\Delta}$. This gives

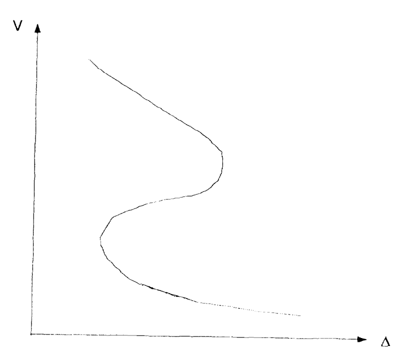

FIG. 2. The speed V of the tip of the fractured zone as a function of Δ, the distance between it and the tip of the finger front. We may also regard Δ as an inverse measure of the stress in the medium ahead of the fractured zone. The lower branch represents a slow, viscous rearrangement of the medium, while the upper branch represents rapid, elastic fracture.

$$\sigma\hat{\Delta} = \hat{V}/A(\Delta_0)\,, \tag{5}$$

$$\tau\sigma\hat{V} = f'(\Delta_0)\hat{\Delta} - \hat{V}\,, \tag{6}$$

where the prime represents differentiation. Eliminating $\hat{V}$ leads to the stability equation for σ,

$$\tau\sigma^2 + \sigma - f'(\Delta_0)/A(\Delta_0) = 0\,. \tag{7}$$

Since $A(\Delta)$ is always positive, the stability is controlled by the sign of $f'(\Delta_0)$: If it is negative, then both solutions of (7) have negative real parts, so that the fixed point is stable. However, if V_0 lies between the two branches of $f(\Delta)$, then there is no fixed point. This could be remedied by introducing a section of $f(\Delta)$ connecting the two branches; intermediate values of V_0 would lead to steady states on this upward-sloping section, which would then be unstable because $f'(\Delta)$ is positive. Even this, however, fails to describe the *onset* of instability as V_0 is increased to this range, since as we make this change, $f'(\Delta)$ becomes infinite.

If the dependence of V on Δ is given by a Z-shaped curve, as in Fig. 2, then $f(\Delta)$ is multiple valued, and Eq. (4) is therefore meaningless. We must then replace it by an equation of the form

$$\tau dV/dt = -g(V,\Delta)\,, \tag{8}$$

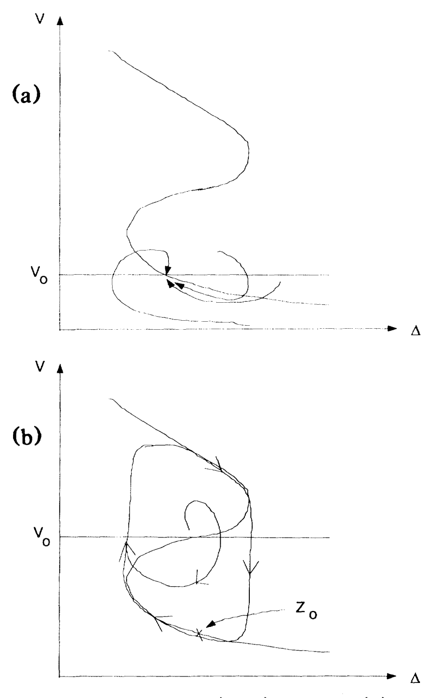

FIG. 3. Trajectories in the (V,Δ) plane. (a) For small V_0, the single fixed point is stable. This represents ordinary Saffman-Taylor fingering, where the medium undergoes slow viscous rearrangement in order to accommodate the slowly advancing finger. (b) For larger V_0, the attractor is a limit cycle. In this cycle, the material alternately undergoes viscous rearrangement while the finger advances and stress builds up, then fractures and releases the stress.

where $g(V,\Delta)$ has two crucial properties. First, it vanishes for $\Delta = F(V)$, where $F(V)$ is the inverse function of $f(\Delta)$. Note that while $f(\Delta)$ is multiple valued, $F(V)$ is a single-valued function. Hence, we have

$$g(V,\Delta = F(V)) = 0\,. \tag{9}$$

Second, V must increase with time for $\Delta < F(V)$ and decrease for $\Delta > F(V)$. Thus $g(V,\Delta)$ must be an increasing function of Δ at $\Delta = F(V)$. Differentiating (9) with respect to V gives

$$(\partial g/\partial V)_{\Delta=F(V)} + F'(V)(\partial g/\partial\Delta)_{\Delta=F(V)} = 0\,, \tag{10}$$

a relation which will soon prove useful. Since $\partial g/\partial\Delta$ is positive in (10), we find that $\partial g/\partial V = -(\partial g/\partial\Delta)F'(V)$ has the opposite sign as $F'(V)$, i.e., as $df/d\Delta$.

Now let us redo the linear stability analysis for (2) and (8), requiring only that g have the minimal properties discussed above. The fixed point has $V = V_0$ and $\Delta = \Delta_0 \equiv F(V_0)$. Again setting $V = V_0 + \hat{V}e^{\sigma t}$ and $\Delta = \Delta_0 + \hat{\Delta}e^{\sigma t}$ and linearizing gives

$$\sigma\hat{\Delta} = \hat{V}/A(\Delta_0)\,, \tag{11}$$

$$\begin{aligned}\tau\sigma\hat{V} &= -(\partial g/\partial V)\hat{V} - (\partial g/\partial\Delta)\hat{\Delta}\\ &= (\partial g/\partial\Delta)[F'(V_0)\hat{V} - \hat{\Delta}]\,,\end{aligned} \tag{12}$$

where the partial derivatives are evaluated at (V_0,Δ_0). Putting (11) and (12) together, we find the quadratic equation for σ,

$$\tau\sigma^2 - (\partial g/\partial\Delta)F'(V_0)\sigma + \frac{\partial g/\partial\Delta}{A(\Delta_0)} = 0\,, \tag{13}$$

whose solutions are given by

$$\sigma = \frac{1}{2\tau}\{(\partial g/\partial\Delta)F'(V_0) \pm \sqrt{(\partial g/\partial\Delta)^2[F'(V_0)]^2 - 4\tau(\partial g/\partial\Delta)/A(\Delta_0)}\}\,. \tag{14}$$

Since both $A(\Delta)$ and $\partial g/\partial\Delta$ are positive, this displays the hallmarks of a Hopf bifurcation [12] as we increase V_0 from the region where $F'(V_0) < 0$ to the region where $F'(V_0) > 0$: σ has an imaginary part, and a real part which changes sign from negative to positive.

Our results are most easily seen by examining the Δ-V phase plane, as sketched in Fig. 3. To the left of the curve $\Delta = F(V)$ all trajectories move upward toward larger V, while trajectories to its right move downward. In Fig. 3(a), the input speed V_0 is small enough for the steady state to be stable. All trajectories quickly reach the fixed point. In this regime, fluid is being pumped into the finger slowly enough that the medium is able to rearrange itself by viscous flow in order to accommodate the advancing finger; the system operates on the lower, "viscous rearrangement" branch of the multiple-valued $V = f(\Delta)$ curve. In Fig. 3(b), we have increased V_0 so that the steady state is now linearly unstable. A trajectory which starts close to the fixed point rapidly spirals outward and converges to a limit cycle. If the response time τ is small, then this limit cycle is a classic relaxation oscillation. To trace the physics of the oscillation, let us start at the point z_0, where the limit cycle crosses the curve $V=f_-(\Delta)$. At this point, the medium is undergoing viscous rearrangement, and so the compliant zone is advancing at a low speed V. However, the finger is advancing more rapidly, so the distance Δ between it and the tip of the zone is decreasing. Consequently, stress builds up ahead of the tip of the zone. Eventually, Δ decreases to the value Δ_- at which the stress is so high that fracture begins. The speed V of the zone tip rapidly increases to $f_+(\Delta)$, which is a true fracture regime. As the tip of the zone advances rapidly, the finger continues to advance at a speed close to V_0. Thus the tip of the zone outraces the finger, Δ increases, and the stress ahead of the fractured zone is relieved. When Δ increases to the value Δ_+, the stress is no longer sufficient to maintain fracture, so the speed V rapidly drops to $f_-(\Delta)$ as the medium reverts to viscous rearrangement, thus completing the cycle.

It is clear from Fig. 3 that once the oscillation sets in, it very quickly loses all memory of the unstable steady state. If τ is small, then the transitions between the upper and lower branches of $f(\Delta)$ happen rapidly, so the period of the oscillation is dominated by the motion of the phase point along these branches. Thus to leading order in τ, the period T is given by

$$T=\int_{\Delta_-}^{\Delta_+}\left[\frac{1}{f_+(\Delta)-V_0}+\frac{1}{V_0-f_-(\Delta)}\right]A(\Delta)d\Delta . \qquad (15)$$

If we increase the rate V_0 at which fluid is being pumped into the system, then the period changes:

$$dT/dV_0=\int_{\Delta_-}^{\Delta_+}\frac{f_+(\Delta)-f_-(\Delta)}{[f_+(\Delta)-V_0]^2[V_0-f_-(\Delta)]^2} \times[2V_0-f_+(\Delta)-f_-(\Delta)]A(\Delta)d\Delta . \qquad (16)$$

Near the onset of instability V_0 is close to $f_-(\Delta_-)$, making the integrand negative. Thus as V_0 is increased from the onset of oscillation, the period of the oscillation *decreases* until V_0 reaches a rather high value, about the average of the viscous and fracture speeds of the compliant zone tip, for which the period T might become solely a material constant, independent of V_0 as in dynamic fracture [13]. In the context of the peeling of an adhesive tape, one literally hears this behavior.

In conclusion, we have posited a simple model for Saffman-Taylor fracture, and found that its dynamics are controlled by a threshold, and that they show a self-sustained oscillation, as is generic in threshold-induced dynamics. Our model shows a sharp transition from a regime dominated by viscous flow of the viscoelastic medium to one characterized by periodic elastic fracture, as observed by Lemaire *et al.* [1]. This transition takes place as the rate at which the inviscid fluid is pumped into the finger is increased past a critical value, i.e., as the time scale for advance of the finger becomes shorter than the time scale for viscous relaxation of the medium. In the fracture regime, the tip of the fractured zone of the medium advances in an oscillatory fashion [9], as stress alternately builds up during a period of insufficiently rapid viscous rearrangement and is relieved by fracture. We find that in order to obtain this behavior, our model *must* include a "bistable" range of stress, which is sufficient to maintain fracture once it has begun, but not sufficient to initiate it. Notice that the signature of the fracture instability in our model is the onset of a self-sustained oscillation in the tip velocity, which could be observed experimentally. Moreover, we predict that once the oscillation has set in, its period should decrease as the driving velocity V_0 is increased.

We wish to thank C. Lobb, J. Maher, and H. Van Damme for discussion of their experiments as well as R. Blumenfeld for discussion for stick-slip dynamics and D. Ou-Yang for discussion on dynamics of peeling. This work is partially supported by the National Science Foundation through the EPSCoR program administered by ASEND in North Dakota (D.A.K.), the Petroleum Research Fund (D.C.H.), and the NSF-IUCRC program through the Center for Polymer Science and Engineering (D.C.H.).

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