# Weak Universality in Two-Dimensional Transitions to Chaos

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Two rational numbers are associated to each periodic orbit of Smale's horseshoe, and practical algorithms are given for their calculation. Using these quantities, it is possible to decide in many cases whether or not a given orbit must always be created after some other given orbit in any two-dimensional transition to chaos. A statement of "weak universality" for the bifurcation sequence as a whole is formulated.

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One of the main endeavors of bifurcation theory is to understand the way in which horseshoes [1] are created. Several authors have studied sequences of bifurcations of periodic orbits [2] in families of homeomorphisms of the plane [3] leading to the formation of a horseshoe [4–10]: They have considered both the necessary similarities and the possible differences between these sequences and the universal bifurcation sequence which occurs for families of unimodal maps of the line (as described by the kneading theory of Milnor and Thurston [11]).

Here we discuss some constraints on the bifurcation sequences of horseshoe-type orbits which are applicable to arbitrary two-dimensional families: Because any $C^{1+\epsilon}$ diffeomorphism of a surface with positive topological entropy has a horseshoe in some iterate [12], our results can be seen as having implications for the mechanism of two-dimensional transitions to chaos in a quite general context. We address two questions, one of a local nature and the other of a global nature. First, given two particular orbits $R$ and $S$ of the horseshoe, is there one of them which is always created before the other when a horseshoe is formed? Second, how strong are the constraints on the order in which orbits can be created in the formation of a horseshoe? We shall present practical algorithms for the calculation of two numbers associated to each horseshoe orbit $R$, the *height* $q(R)$ and the *depth* $r(R)$, which enable us to answer the first question in many cases (for any horseshoe orbits $R$ and $S$). (1) If $r(R) < q(S)$ then in any family $\{f_\mu\}_{\mu\in[0,1]}$ of homeomorphisms for which $f_0$ has trivial dynamics and $f_1$ has a horseshoe, $R$ is created after $S$. (2) If $q(R) > q(S)$ or $r(R) > r(S)$, then there exists a family $\{f_\mu\}$ in which $R$ is created before $S$. In particular, if $q(R) > q(S)$ and $r(S) > r(R)$, then either $R$ or $S$ can be created first.

A precise statement is given later (Theorem 1). This result can be applied to the second (global) question to formulate a statement of *weak universality* for two-dimensional transitions to chaos (Theorem 2), which is a weak analog of the topological universality which exists in the one-dimensional context. We shall also explain how our methods enable us to obtain a lower bound on the topological entropy of a partially formed horseshoe, given information about one of its orbits.

Let us choose a model for the horseshoe $f$ which takes a square $B$ in the plane, expands it uniformly in the horizontal direction, contracts it uniformly in the vertical direction, and then maps the left half of the deformed square across $B$ preserving orientation, and the right half across $B$ reversing orientation. We introduce symbolic dynamics for $f$ in the usual way [13]: Each point $x$ in the nonwandering set $\Omega(f)$ has an associated *itinerary*, which is the bi-infinite sequence $c(x) = \ldots c_{-1}(x)c_0(x) \times c_1(x)\ldots$, where $c_i(x) = 0$ if $f^i(x)$ lies in the left half of the square $B$, and $c_i(x) = 1$ if $f^i(x)$ lies in the right half. The horizontal ordering of points in $\Omega(f)$ is then reflected by the *unimodal* order on semi-infinite symbol sequences [13]: If $s$ and $t$ are two such sequences which first differ on the $n$th symbol, we say that $s \succ t$ if and only if the first $n$ symbols of $s$ contain an odd number of 1's. Then for any two points $x$ and $y$ in $\Omega(f)$, $x$ lies to the right of $y$ in $B$ if and only if $c_0(x)c_1(x)\ldots \succ c_0(y)c_1(y)\ldots$. If $R$ is a period $n$ orbit of the horseshoe, then its *code* $c_R$ is defined to be $c_0(x)\ldots c_{n-1}(x)$, where $x$ is the point of $R$ which lies furthest to the right in $B$ [thus, for example, the orbit $R$ which contains the point of itinerary $(01111010)^\infty$ has code $c_R = 10011110$]. We shall write $c_R^\infty$ for the semi-infinite sequence $c_Rc_Rc_R\ldots$, and ${}^\infty c_R^\infty$ for the bi-infinite sequence $\ldots c_Rc_Rc_R\ldots$ (which is the itinerary of $x$).

An important observation is that two or more distinct horseshoe orbits may be topologically indistinguishable. For example, the period 7 orbits $R$ and $S$ with codes $c_R = 1001011$ and $c_S = 1001110$ are so: They can be distinguished by their codes in the full horseshoe $f$, but there is no topological content to the statement that a particular orbit of a partially formed horseshoe "is" $R$ rather than $S$ (it is possible to find a family $\{f_t\}$ of homeomorphisms of the plane, where $f_0$ and $f_1$ are full horseshoes and each $f_t$ is horseshoelike, such that when the orbit $R$ of $f_0$ is followed through the family it ends up as the orbit $S$ of $f_1$ [14]). In general, orbits $P$ and $Q$ of homeomorphisms $g$ and $h$ are topologically indistinguishable if there is a family $\{g_t\}$ of homeomorphisms, with $g_0 = g$ and $g_1 = h$, such that the orbit $P$ can be followed continuously


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through the family to end up as the orbit $Q$. When this is the case, we shall say that $(P,g)$ and $(Q,h)$ *have the same braid type* [15] (the name is chosen because when a homeomorphism arises as a return map for a flow in three dimensions, the braid type of a periodic orbit $P$ is closely related to the braid which is formed by $P$ regarded as a periodic orbit of the flow [15]). In the above example, $(R,f)$ and $(S,f)$ have the same braid type: If $P$ is an orbit of a partially formed horseshoe $g$, then all we can meaningfully say is that the braid type of $(P,g)$ is the same as that of both $(R,f)$ and $(S,f)$. With this in mind, we say that a function defined on the set of all horseshoe orbits is a *braid-type invariant* if it takes the same value on any two orbits which have the same braid type: This is necessary if the function is also to be well defined for orbits of partially formed horseshoes.

We shall begin by presenting the height and depth algorithms, and giving statements of our main results. We shall limit ourselves to an explanation of how the height and depth arise; full details, including derivations of the algorithms, can be found elsewhere [16] (the proofs depend on the Nielsen-Thurston classification of isotopy classes of surface homeomorphisms [17]). The main principle underlying the results is that the existence of a single orbit of a dynamical system can force the coexistence of many other orbits. In the field of discrete dynamics, the first example of this principle was provided by Šarkovskiĭ's theorem, which describes the possible sets of periods which can occur for continuous maps of the real line in terms of a "forcing order" on the positive integers [18]. This idea was later generalized to define other forcing orders for one- and two-dimensional systems. However, an orbit in a system of three or more dimensions does not give enough information to infer the existence of other orbits: For this reason, it seems unlikely that there are comparable statements of universality for transitions to chaos in higher dimensions.

Let us describe the algorithms for determining the height and depth of a horseshoe orbit. These are rational numbers lying between 0 and $\frac{1}{2}$; the algorithms used for calculating them are both expressed in terms of a primitive process, which associates a number $q(c) \in (0,1/2]$ to each sequence $c$ which begins $c=10\ldots$ and which contains the group of symbols $\ldots010\ldots$, as follows: Write $c$ in the form $c=10^{\kappa_1}1^{\mu_1}0^{\kappa_2}1^{\mu_2}\ldots$, where $\kappa_i \geq 0$ and $\mu_i$ is either 1 or 2 for each $i$, with $\mu_i=1$ only if $\kappa_{i+1}>0$ (thus $\kappa_i$ and $\mu_i$ are uniquely determined by $c$, and the fact that $c$ contains the group $\ldots010\ldots$ means that $\mu_s=1$ for some $s$). Let

$$I_r(c) = \left( \frac{r}{2r+\sum_{i=1}^{r}\kappa_i}, \frac{r}{(2r-1)+\sum_{i=1}^{r}\kappa_i} \right]$$

for each $r \geq 1$, and let $s$ be the least integer such that either $\mu_s=1$ or $\cap_{i=1}^{s+1} I_i(c)=\varnothing$. Then $\cap_{i=1}^{s} I_i(c)=(x,y]$ for some $x$ and $y$: We define $q(c)=y$ if $\mu_s=1$ or $I_{s+1}(c) > \cap_{i=1}^{s} I_i(c)$; and $q(c)=x$ if $\mu_s=2$ and $I_{s+1}(c) < \cap_{i=1}^{s} I_i(c)$.

*Definitions.*—Let $R$ be a horseshoe orbit. We define $\bar{c}_R$ to be $c_R$ if $c_R^\infty$ contains the group of symbols $\ldots010\ldots$, and to be the code obtained by changing the final symbol of $c_R$ from 1 to 0 otherwise. The *height* $q(R)$ of $R$ is the number $q(\bar{c}_R^\infty)$. The *depth* $r(R)$ of $R$ is defined as follows: Consider all strings of the form $\ldots01110\ldots$ or $\ldots01010\ldots$ in the sequence ${}^\infty\bar{c}_R^\infty$; suppose that there are $l$ such strings $g_1,\ldots,g_l$ contained in one period of ${}^\infty\bar{c}_R^\infty$. If $l=0$ then $r(R)=\frac{1}{2}$. Otherwise, for each $i \leq l$ let $f_i$ be the code obtained by starting at the last 1 in $g_i$ and moving forwards through ${}^\infty\bar{c}_R^\infty$, and $b_i$ be that obtained by starting at the first 1 and moving backwards; let $M_i=\max[q(f_i),q(b_i)]$. Then $r(R)=\min_{1\leq i\leq l} M_i$.

Our first main result makes precise the earlier informal statement.

*Theorem 1.*—Let $R$ and $S$ be orbits of the horseshoe map $f$. (1) If $r(R)<q(S)$ then every homeomorphism which has an orbit of braid type $b(R,f)$ has at least as many orbits of braid type $b(S,f)$ as does the horseshoe. (2) If $q(R)>q(S)$ or $r(R)>r(S)$, then there exists a homeomorphism which has an orbit of braid type $b(R,f)$ but none of braid type $b(S,f)$.

*Examples.*—Let $R$ be the period 22 orbit with code $c_R=\bar{c}_R=1000011000110000110010$. Then $\kappa_1=4$, $\mu_1=2$, $\kappa_2=3$, $\mu_2=2$, $\kappa_3=4$, $\mu_3=2$, $\kappa_4=2$, and $\mu_4=1$. Thus $I_1=(\frac{1}{6},\frac{1}{5}]$, $I_2=(\frac{2}{11},\frac{2}{10}]$, $I_3=(\frac{3}{17},\frac{3}{16}]$, and $I_4=(\frac{4}{21},\frac{4}{20}]$. Since $\frac{4}{21}>\frac{3}{16}$ we have $I_1\cap I_2\cap I_3\cap I_4=\varnothing$, and $I_4>I_1\cap I_2\cap I_3$. Therefore $q(R)=\max(I_1\cap I_2\cap I_3)=\frac{3}{16}$. There is a single string $\ldots01010\ldots$, with $q(f_1)=\frac{3}{16}$ and $q(b_1)=q(100110000\ldots)=\frac{1}{4}$. Thus $r(R)=\frac{1}{4}$. Similarly, the orbit $S$ with code $c_S=10111101010$ has $q(S)=\frac{3}{7}$ and $r(S)=\frac{1}{2}$, and the orbit $T$ with code $c_T=100001110000$ has $q(T)=r(T)=\frac{1}{5}$. The fact that $r(R)<q(S)$ implies that in any family of homeomorphisms leading to the creation of a horseshoe, $S$ must be created before $R$, and the fact that $q(R)<q(T)=r(T)<r(R)$ means that there exist families in which $R$ is created before $T$, and other families in which $T$ is created before $R$.

The height and the depth are both braid-type invariants. In fact, there is a relationship between height and a well-established braid-type invariant. Orbits of the horseshoe have rotation numbers lying between 0 and $\frac{1}{2}$ about the fixed point with code 1; for each orbit $R$, consider the set of rotation numbers of all those orbits which must be created before $R$ in any transition to chaos: This set is a (possibly degenerate) closed interval in the rationals, called the *rotation interval* $\rho i(R)$ of $R$. The height of a horseshoe orbit is equal to the lower end point of its rotation interval. A proof of this is given in Ref. [16], where a practical algorithm for determining $\rho i(R)$ is described. The other end point of the rotation interval is a less interesting quantity: It is equal to $\frac{1}{2}$ for "most" horseshoe orbits. To be precise, the higher end point of

$pi(R)$ is $\frac{1}{2}$ if and only if $c_R^\infty$ contains a string either of the form $\ldots 01010 \ldots$ or of the form $\ldots 01^{2k+1}0 \ldots$ for some $k \geq 1$: Clearly the proportion of period $n$ orbits which have this property tends to 1 as $n$ tends to infinity.

Now let us use Theorem 1 to make a more global statement about the order in which orbits can be created. Suppose that $R$ is a horseshoe orbit with height $q(R)=q$. Then by Theorem 1, any orbit $S$ with $r(S)<q$ has the property that it must be created after $R$ in any family of homeomorphisms leading to the formation of a horseshoe. However, the proportion of period $n$ orbits with this property tends to 1 as $n$ tends to infinity, since if $c_S$ contains the string of symbols $\ldots 0^{k-1}1110^{k-1} \ldots$ then $r(S) \leq 1/k$.

*Theorem 2 (weak universality).*—Let $R$ be a horseshoe orbit, and for each $n \geq 1$ let $p_n(R)$ be the proportion of period $n$ horseshoe orbits $S$ such that any homeomorphism with an orbit of braid type $b(S,f)$ has as many orbits of braid type $b(R,f)$ as does the horseshoe. Then $p_n(R) \to 1$ as $n \to \infty$.

Constraints on the order of creation of orbits of low period can be determined exactly using train track methods [19]. However, these methods are impractical for periods much greater than 10, whereas the algorithms we have described are easy to implement for orbits of period 10000 or more (notice that the time required to determine the height and depth of a period $n$ orbit $R$ given its code $c_R$ is linear in $n$). Figure 1 gives full constraints for the orbits of period 7 and less: an orbit $R$ must always be created after an orbit $S$ (that is, $R \geq_2 S$) if and only if there is a path from $R$ to $S$ in the diagram. The nomenclature is as follows: Consider a list of period $n$ orbits, arranged with increasing codes in the unimodal order. Two orbits whose codes differ only in their last symbol are always adjacent in this order, and are known to have the same braid type: Since it is meaningless to distinguish orbits of the same braid type in such a diagram, such pairs of orbits are identified. Then the orbit labeled $n_k$ is the $k$th orbit in the list (thus $4_1$ denotes the orbit of code 1011, and $4_2$ the pair of orbits with codes 1001 and 1000). Since the orbits $7_3$ and $7_4$ (for example) also have the same braid type, they have been paired in the diagram. The fixed point $1_1$ has been omitted: It is always the first orbit to be created.

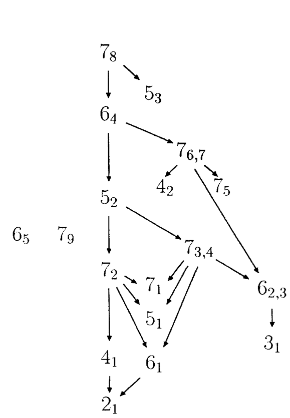


FIG. 1. Bifurcational precedences for low-period horseshoe orbits.

The height and depth arise from a consideration of those features of the universal one-dimensional bifurcation sequence which must always be reproduced in two dimensions. The two situations can be compared because the one-dimensional map obtained from the horseshoe by collapsing along (vertical) stable leaves is unimodal: Thus there is a symbolic dynamics for unimodal maps corresponding to that for the horseshoe. To each orbit $R$ of the horseshoe, we associate the orbit of the unimodal map which has the same code, which we also denote $R$. Given a pair $R,S$ of horseshoe orbits, we write $R \geq_2 S$ if $R$ is always created after $S$ in any family of homeomorphisms of the plane, and we write $R \geq_1 S$ if the one-dimensional orbit $R$ is always created after the one-dimensional orbit $S$ in any family of maps of the line. The fact that there is a universal order in which the orbits are created in the one-dimensional context is equivalent to saying that for every such pair, either $R \geq_1 S$ or $S \geq_1 R$. This is not true in two dimensions: For instance, the two orbits $R$ and $T$ given in the examples after Theorem 1 have $R \not\geq_2 T$ and $T \not\geq_2 R$.

Given $R$ and $S$, it is easy to determine whether or not $R \geq_1 S$ using kneading theory (in fact, if $R$ and $S$ are distinct then $R \geq_1 S$ if and only if $c_R^\infty > c_S^\infty$ [20]); the problem of determining whether or not $R \geq_2 S$ is much harder. This leads us to consider the set of orbits $R$ for which $R \geq_1 S$ implies that $R \geq_2 S$. Such orbits will be called *quasi-one-dimensional* (qod): They are relatively rare, since the constraints on the one-dimensional bifurcation sequence are stronger than those on the two-dimensional bifurcation sequence. The fact which enables us to define the height and depth is that for each rational $q$ with $0<q<\frac{1}{2}$, there is a corresponding qod orbit $P_q$, and that these orbits have the property that $P_q \geq_2 P_{q'}$ if and only if $q \leq q'$. Thus in any family of homeomorphisms leading to the creation of a horseshoe, the orbits $P_q$ appear in order of descending $q$. The height $q(R)$ and the depth $r(R)$ of an orbit $R$ are the limiting values of $q$ which tell us, respectively, the latest and earliest that $R$ can be created relative to the orbits $P_q$ [thus $P_q \geq_2 R$ for $q<q(R)$, while $P_q \not\geq_2 R$ for $q>q(R)$; and $R \geq_2 P_q$ for $q>r(R)$, while $R \not\geq_2 P_q$ for $q<r(R)$]. It is easy to see how Theorem 1 follows from these descriptions. For example, if $r(R)<q(S)$, let $q$ be a rational number with $r(R)<q<q(S)$: Then $R \geq_2 P_q$ and $P_q \geq_2 S$, so that $R \geq_2 S$.

The orbits $P_q$ are described as follows: If $q=m/n$ in lowest terms, then $P_q$ has period $n+2$, and its code is $c_q = 10^{\kappa_1(q)}1^2 0^{\kappa_2(q)}1^2 \ldots 1^2 0^{\kappa_m(q)}10$, where $\kappa_1(q) = \lfloor 1/q \rfloor - 1$ and $\kappa_i(q) = \lfloor i/q \rfloor - \lfloor (i-1)/q \rfloor - 2$ for $2 \leq i \leq m$ (here $\lfloor x \rfloor$ denotes the greatest integer which does not

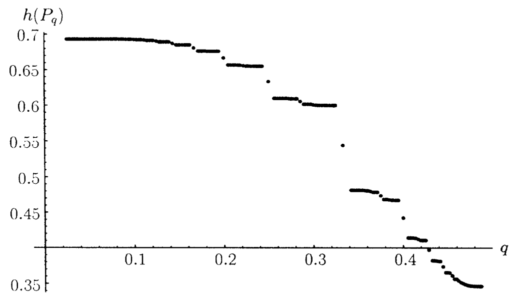


FIG. 2. Entropy of the quasi-one-dimensional orbits $P_q$.

exceed $x$). Thus, for example, we have $c_{1/3}=10010$, $c_{1/4}=100010$, $c_{1/5}=1000010$, and $c_{2/5}=1011010$. (In fact, there is a mathematical characterization of the family of orbits $\{P_q\}$: They are exactly the qod orbits which have pseudo Anosov braid type [15].) The reader familiar with the work of Holmes and Williams on knot types of orbits in the suspension of the horseshoe [8] will notice that the first $n$ symbols of the code of the period $n+2$ orbit $P_{m/n}$ give the code of an orbit which defines an $(m,n)$ resonant torus knot. The author has no explanation for this observation.

We finish by indicating how to obtain lower bounds for the topological entropy of a partially formed horseshoe. Given a horseshoe orbit $R$, let $h(R)$ denote the smallest possible entropy of a homeomorphism having an orbit of the same braid type as $R$. Now we know that for all $q>r(R)$ we have $R \geq {}_2P_q$, and hence $h(R) \geq h(P_q)$. However, since $P_q$ is a qod orbit, it can be shown that $h(P_q)$ is equal to the entropy of $P_q$ regarded as an orbit in one dimension: This can rapidly be calculated using standard transition matrix techniques. A graph of the function $q \mapsto h(P_q)$ is given in Fig. 2: It is discontinuous everywhere. Thus if $g$ is a partially formed horseshoe which has an orbit with the braid type of $R$, then $h(g) \geq h(P_q)$ for all $q>r(R)$. For example, the orbit $R$ with code $c_R=10011010$ has $r(R)=\frac{1}{3}$: Thus any partially formed horseshoe which includes an orbit of the braid type of $R$ has entropy greater than $h(P_{2/5}) \simeq 0.442$. Better bounds can be obtained by picking rationals closer to $\frac{1}{3}$: For example, $h(P_{5/14}) \simeq 0.481$. Notice that because the value of $r(R)$ depends only on local properties of the code of $R$, this method provides bounds on the entropy which depend only on small segments of the code of an orbit: For example, if $g$ has an orbit $S$ whose code contains the string $\ldots 0011100 \ldots$, then $r(S) \leq \frac{1}{3}$, and hence $h(g) > 0.481$.

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