Quantum Signatures of Classical Chaos: Sensitivity of Wave Functions to Perturbations

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We compare the influence of perturbations of classically regular respectively chaotic Hamiltonians on their eigenfunctions. A generic measure of the perturbation strength is given by the average spreading width, which semiclassically is a phase-space integral. Thus, contrary to common assumption, the spreading width cannot be an indicator of regularity or chaos. The distribution of expansion coefficients of perturbed eigenfunctions in terms of unperturbed ones is markedly different for the two cases and may serve as a quantum signature of chaos referring to states rather than to spectra.

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What are generic quantum signatures of classical chaos? It was recognized about ten years ago [1,2] that the distribution of eigenvalues is a suitable candidate, and it is now established [3] that this distribution generically has GOE-like fluctuations for classically (fully) chaotic systems, and Poisson-like fluctuations for classically integrable systems. (Here GOE stands for the Gaussian orthogonal ensemble of random matrices [4].)

In the case of eigenfunctions, despite early efforts (e.g., [5–8]), a similarly well-defined signature is not yet known. Many properties of wave functions have been discussed but they are usually difficult to express in a basis-independent form (e.g., [8–10]), in spite of some general arguments for the ergodicity of wave functions [11].

Zaslavsky [12] conjectured that eigenfunctions of classically chaotic systems may be more sensitive to changes δH of the Hamiltonian than those of classically integrable ones. Inspection shows that this conjecture must be modified, but serves nonetheless as a useful point of departure. This is because such a sensitivity is essentially basis independent as it is naturally expressed in the eigenbasis of the unperturbed Hamiltonian.

In the present Letter, we analyze the question of sensitivity to perturbations in depth. First we introduce a universal measure for the strength of a perturbation, then scrutinize Zaslavsky's conjecture, present results obtained for a particular model system, and are finally led to a quantification of Zaslavsky's idea.

A comparison of the sensitivity of wave functions to perturbations requires a definition of the strength of such perturbations that does not depend on details of the dynamics. In defining the strength of δH, we are guided by the concept of the spreading width [13]. We consider an unperturbed Hamiltonian matrix H of dimension N with eigenvalues E_i, $i=1,\dots,N$, orthonormal eigenvectors Φ_i, $i=1,\dots,N$, and an average level density $\rho(E)$ at energy E. The spreading width $\Gamma_i^{\downarrow}$, a measure of the strength of the perturbation δH in its action on the state Φ_i, is defined as

$$\Gamma_i^{\downarrow}=\frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N}|\langle\Phi_i|\delta H|\Phi_j\rangle|^2\rho(E_i) \tag{1a}$$

$$\approx\frac{1}{N}\sum_{j=1} N|\langle\Phi_i|\delta H|\Phi_j\rangle|^2\rho(E_i) \tag{1b}$$

$$=\frac{1}{N}\langle\phi_i|(\delta H)^2|\Phi_i\rangle\rho(E_i)\,. \tag{1c}$$

Here, Eq. (1a) defines $\Gamma_i^{\downarrow}$, Eq. (1b) holds generically (in particular if the matrix δH is not dominated by the diagonal elements) for $N\gg 1$, and Eq. (1c) follows from completeness. We note that $\Gamma_i^{\downarrow}$ measures the energy interval over which δH effectively spreads the eigenfunction Φ_i of H. Clearly, we are interested only in the *average* (over a group of neighboring states Φ_i) of $\Gamma_i^{\downarrow}$. Let $P=(1/n)\times\sum_{i=1}^{n}|\Phi_i\rangle\langle\Phi_i|$ be the normalized projector onto the corresponding subspace. Assuming that $\rho(E)$ does not change significantly in the energy interval defined by the n eigenvalues of H pertaining to P, we have

$$\Gamma^{\downarrow}=\overline{\Gamma_i^{\downarrow}}=\mathrm{tr}\{P\delta H^2\}\rho/N\,. \tag{2}$$

We note that the factors appearing on the right-hand side of Eq. (2) each have a well-defined classical limit. The first can be written as

$$\mathrm{tr}\{P\delta H^2\}\approx\left(\frac{1}{2\pi\hbar}\right)^f\int_V dp^f dq^f(\delta H)^2\Big/\left[\left(\frac{1}{2\pi\hbar}\right)^f\int_V dp^f dq^f\right]. \tag{3}$$

Here f is the number of degrees of freedom, and V is that part of phase space which corresponds to the projector P. (Strictly speaking, we should employ the Wigner transform of P in lieu of V but suppress this complication.) The remaining factors can likewise be written as phase-space integrals, but phase-space integrals are insensitive to dynamical features of the Hamiltonian and in particular to integrability or chaos. Thus we have found a universal measure for the strength of perturbations if the integer n is chosen large enough to guarantee that in either case (regular or chaotic)

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comparable parts of phase space are covered by the Wigner transform of P.

At this point we have not only defined a measure for the strength of the perturbation, we have also implicitly disproved Zaslavsky's conjecture in its most naive form. Indeed, the spreading width is perhaps the most natural measure of the sensitivity of wave functions to perturbation, and it is obviously not sensitive to classical chaos.

Although we shall not consider small perturbations, it is helpful to recall what perturbation theory would make us expect. The dominant factor at very small level spacings is the energy denominator $(E_i - E_j)^{-1}$. Since small level spacings are much more frequent for integrable systems than for chaotic ones, we will see strong mixing for integrable systems situations. Hence, a refinement of the argument is needed. It is supplied by Percival's remark [5] that, in the integrable case, only states with similar quantum numbers are expected to be mixed strongly by a perturbation, while in the chaotic case, the mixing is expected to be much more uniform. We show below that this argument is qualitatively correct, and that it is the uniformity rather than the degree of mixing of the eigenfunctions which distinguishes regular and chaotic systems.

We thus have to consider the uniformity of the mixing of unperturbed eigenfunctions by δH. For this purpose, we take the expansion of the orthonormal eigenfunctions Ψ_μ of the full Hamiltonian $H+\delta H$ in terms of the unperturbed eigenfunctions Φ_i of H,

$$\Psi_\mu = \sum_k a_{k\mu}\Phi_k \,. \tag{4}$$

Obvious measures of uniformity are the probability distribution of the expansion coefficients and derivative quantities like the average participation ratio $\overline{P^-}$ and the average inverse participation ratio $\overline{P^+}$. The latter two quantities are defined as averages over n neighboring eigenvalues,

$$\overline{P^{\mp}} = \frac{1}{n}\sum_{i=i_0-n/2}^{i_0+n/2}\left[\sum_\mu |a_{i\mu}|^4\right]^{\mp 1}. \tag{5}$$

To compare the participation ratios as given by Eq. (5) with the expression for the spreading width, Eq. (1), the latter is rewritten in terms of the expansion coefficients $a_{k\mu}$ as

$$\Gamma_i^{\downarrow} = \left[\sum_\mu |a_{i\mu}|^2(\mathcal{E}_\mu - E_i)^2\right]\rho(E_i)/N \,. \tag{6}$$

Here, $\mathcal{E}_\mu$ is the eigenvalue of $H+\delta H$ pertaining to Ψ_μ. To get to Eq. (6), we have written δH in Eq. (1a) as the difference between the full and the unperturbed Hamiltonian, and have used the eigenvalue equations for both Hamiltonians.

We test these measures on the following model [14]: We consider two coupled quartic oscillators with the Hamiltonian

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + \alpha_1 q_1^4 + \alpha_2 q_2^4 + \alpha_{12}(q_1 - q_2)^4 \,, \tag{7}$$

in obvious notation and with $\alpha_1, \alpha_2 > 0$. This system is integrable for $\alpha_{12} = 0$ and chaotic when α_{12} is negative and in absolute value close to but below the dissociation limit, $|\alpha_{12}| < \frac{1}{16}(\alpha_1 + \alpha_2)$. The chaotic Hamiltonian was tested by a Monte Carlo calculation and was found to have a chaotic phase-space component of more than 99% [14].

We choose α_1 different from α_2 and note that then H as defined in Eq. (7) is not rotationally symmetric in the (q_1, q_2) plane. In this situation, there is a very convenient way to choose the perturbation δH for both the regular and the chaotic case: Together with the operator H defined in Eq. (7), we consider the operator $H'(\beta)$ obtained from H by a rotation about an angle β as well as reflection on one axis in the (q_1, q_2) plane and define δH as the difference $H - H'(\beta)$. This procedure allows us to attain equal average level density and comparable perturbations in the regular and chaotic cases. For fixed α_{12} (which determines chaotic or regular situations) the product $\alpha_1\alpha_2$ determines the level density, and (for fixed $\alpha_1\alpha_2$) the value of α_1 (and of β) in turn fixes the strength of δH as given by Eqs. (2) and (3). Thus we can simultaneously have equal level density and almost identical perturbation strength for an integrable and a chaotic system as we would like to have for a simple comparison. Also the so-defined perturbation leaves the integrable Hamiltonian integrable and the chaotic one chaotic. These combined considerations led us to choose for the

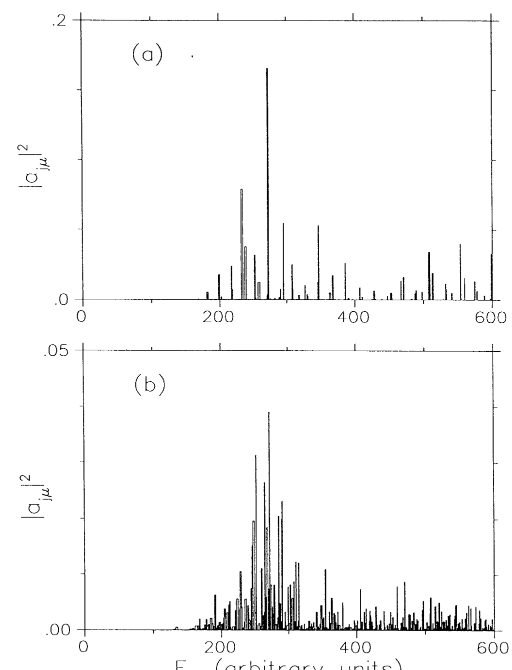

FIG. 1. The intensity spectrum for the (arbitrarily chosen) eigenstate number 152 of the perturbed system is plotted versus the energy eigenvalues of the unperturbed system for both the chaotic and the integrable case over the range of the lowest 600 states. Note the difference in scale.

integrable system $\alpha_1 = 17.8$, $\alpha_2 = 60.955$, and $\alpha_{12} = 0$ and for the chaotic system $\alpha_1 = 35.15$, $\alpha_2 = 76.2$, and $\alpha_{12} = -5.0$, as well as $\beta = 90°$.

Using these parameters we find for the phase-space integral in Eq. (3) at energy $E = 1$ the values 0.4675 and 0.4752 for the integrable and chaotic situations, respectively. Thus we fulfill our criterion for equal perturbation in both cases to very good approximation. (Scaling permits us to consider one energy only.)

Our quantum results are obtained numerically by a variational procedure in a harmonic oscillator basis [14]. Figure 1 shows a typical intensity spectrum: We plot the probability $|a_{j\mu}|^2$ [see Eq. (1)] of finding in some fixed but arbitrary eigenstate Ψ_μ of $H + \delta H$ (here $\mu = 152$) the eigenstates Φ_j of the unperturbed Hamiltonian H versus the eigenvalue E_j of H, for both the integrable and the chaotic system. The difference is striking: Only a few lines dominate the integrable case. (Note the difference in scales.)

We first check whether the spreading widths in both cases are really equal. To this end, we compute the quantity [cf. Eqs. (3) and (6)] $A_i = \sum_\mu |a_{i\mu}|^2 (\mathcal{E}_\mu - E_i)^2$ as a measure of $\Gamma_i^\downarrow$ because the level density was fixed to be equal in both cases. We have calculated the average $\bar{A}$ and $\sqrt{\mathrm{var}(A)}/\bar{A}$ over a range of $2n = 100$ levels around E_{i_0}. The results are shown in Table I for both the integrable and chaotic cases. We note that within 1 or 2 standard deviations, the mean values $\bar{A}$ agree in both cases, giving credence to our prescription. We have similarly calculated the average participation ratios, Eq. (5). In principle, the summation over μ should extend over the entire spectrum. Obviously, we can only sum as far as the expansion of eigenfunctions of the perturbed systems in terms of those of the unperturbed ones is reliable. This would seriously restrict our statistics, but we can avoid the problem by summing only over a window of 200 states and renormalizing the amplitudes. The window is justified by the decay of the peak in Fig. 1.

TABLE I. The averages of A_i related to the spreading width and of the participation ratio are listed for the classically integrable and chaotic systems (indicated by I and C). Averages are taken around different state numbers i_0 and with a window size of $n = 100$ levels. The fluctuations of A as measured by $F(A) = \sqrt{\mathrm{var}(A)}/\bar{A}$ are also shown.

i_0	$\bar{A}$	$F(A)$	$\overline{P^-}$
I 100	2701.0	0.62	9.13
I 200	2227.6	0.59	9.51
I 300	1702.9	0.51	8.96
I 400	1495.2	0.43	8.66
I 500	1370.1	0.29	8.66
C 100	2617.9	0.23	39.0
C 200	2341.0	0.19	49.0
C 300	1793.5	0.20	50.9
C 400	1492.2	0.19	53.5
C 500	1366.9	0.20	55.7

The results are shown for spectra of a given parity and averaged over both parities to improve statistics. Yet the integrable Hamiltonian has an additional symmetry as parity in q_1 and q_2 directions are conserved separately and this symmetry is not broken by our particular perturbation ($\beta = 90°$). This introduces a trivial factor of 2 that divides the $\overline{P^-}$ in the integrable case to have a fair comparison with the chaotic one. We have to keep this factor (not contained in the numbers of the table) in mind when looking at column 4 of Table I where the average participation ratio is shown. Nevertheless the participation ratio is significantly smaller for the integrable case than for the chaotic one.

Next we study the probability distribution of amplitudes. The difficulty here is that systematic trends are superposed on statistical fluctuations; this is clearly visible in Fig. 1. We have applied the following procedure to overcome this obstacle: We took nine levels near the center of the distribution, and plotted in Fig. 2 the frequency distribution with which an amplitude occurs versus the (real) value of this amplitude. For the chaotic case, we find a Gaussian distribution as expected. But for the integrable case, we find a large percentage of very small amplitudes as well as a few very large ones. Indeed

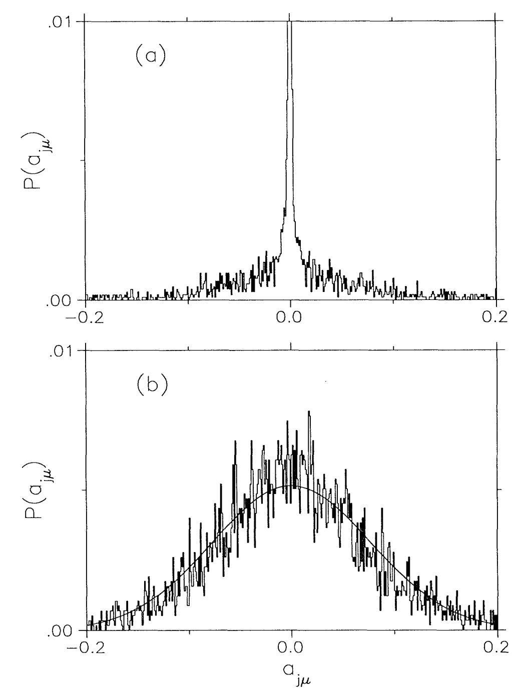

FIG. 2. For the lowest 600 states the frequency of the amplitudes of the perturbed in terms of the unperturbed system are plotted; only the nine amplitudes of the state nearest to the unperturbed state were used. Note that the peak for very small amplitudes was cut in this figure; the two bins exceeding the frame contain 75.2% of all chosen amplitudes.

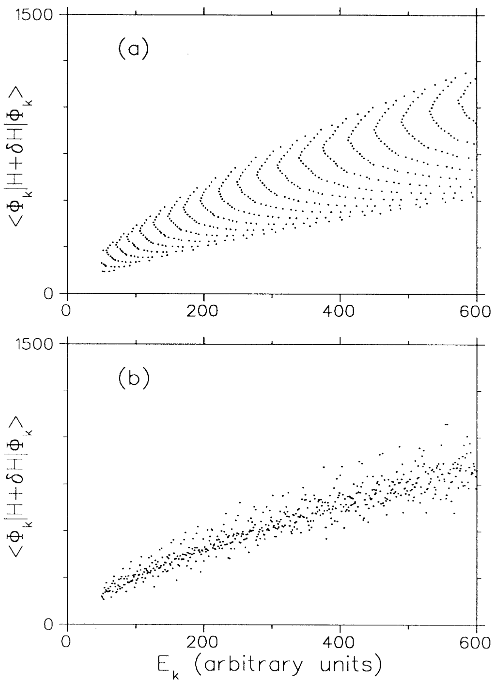

FIG. 3. For eigenstates 50 to 600 of the unperturbed Hamiltonian the expectation values of the perturbed Hamiltonian are shown for both the integrable (a) and the chaotic (b) system. Note the regularities in the integrable case.

in Fig. 2 the probability of finding very small amplitudes is exceeding the frame. The two bins around zero make up about 75.2% of the total probability and even if we reduce it to 25.2% by omitting the zeros due to the above mentioned selection rule this does not qualitatively change the picture.

As an important aside, we would like to draw the readers' attention to a remarkable though unexplained observation in the integrable case. In an earlier paper [15] the participation ratio of a certain matrix element displayed some regular features as a function of the state number, and we found similar behavior in our calculations. Particularly striking results were found for the matrix element of the perturbed Hamiltonian in terms of the unperturbed eigenstates $\langle\Phi_k|H+\delta H|\Phi_k\rangle$. In Fig. 3 we show this matrix element as a function of the unperturbed eigenvalue E_k. A striking difference is apparent between the chaotic and the ordered case. At this point we do not know whether this feature is a peculiar one of the quartic oscillator as the calculations of Ref. [15] were performed on a similar system. Yet it is interesting because it could be associated with Percival's [5] original description of *close* states in the integrable case.

Note that with respect to this figure as well as for all previous results a perturbation that converts the ordered system into a disordered one behaves essentially the same way as for an initially disordered system. Also similar results are found for other angles β.

In conclusion, we have established the following results: (i) We have shown that the spreading width is semiclassically given by a phase-space integral. This integral is of a type similar to the one for the level density and does not contain information about the (chaotic or regular) dynamics of the system. This disproves Zaslavsky's conjecture in its simple-minded form. (ii) By its very nature as a phase-space integral, the spreading width is a useful measure of the strength of the perturbation. This measure enables us to compare regular and chaotic cases in a meaningful way. (iii) Using the results (i) and (ii), we find a qualitatively different behavior of a regular and a chaotic system under the influence of a perturbation as seen in Fig. 2. The very obvious difference can be quantified by taking moments of this distribution.

It is desirable to establish the domain of universality of the features reported in this Letter, and to obtain a quantitative relationship between $\overline{P^{\pm}}$ and the spreading width for the regular and chaotic cases. We hope to return to these issues in a future publication.

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