Characterization of Spatiotemporal Chaos from Time Series

M. Bauer,* H. Heng, and W. Martienssen
Physikalisches Institut der Universität Frankfurt, Robert-Mayer-Strasse 2-4, D-6000 Frankfurt-1, Federal Republic of Germany
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The analysis of time series generated by spatiotemporal chaotic systems is discussed. We find that a Grassberger-Procaccia algorithm with a suitable normalization for the correction of systematic errors caused by the shape of the attractor is a reliable method for distinguishing between high-dimensional chaos and noise. We show that even a quantitative description of the attractor is possible by means of dimension densities. The results obtained from the time series are in good agreement with values calculated directly from the generating equations of motion.

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Computer based methods for the analysis of time signals have become an important tool to characterize the dynamics of deterministic chaotic systems [1]. In this context, the most established method is the Grassberger-Procaccia algorithm (GPA) [2,3], which provides the fractal dimension D_2 of the attractor. It is a common feature of such algorithms that the number of required data points increases exponentially with the dimension of the attractor [4], resulting in exponentially growing requirements on measuring time, computer time, and data storage capacity. Because of this fact the GPA has been applied in most cases only to low-dimensional (typically $D_2 \lesssim 5$) systems.

However, there is a large class of physical and technical systems which cannot be described by low-dimensional attractors. Typical examples are spatially extended systems showing chaotic behavior in time and space [5]. The dimension D of the chaotic attractor grows with the size of the system and it is not unusual to find values for D much larger than 100. In those cases it is impossible to calculate the dimension D_2 directly by applying the GPA to the attractor of the system.

Because of this fact it is generally believed that the restrictions of the GPA exclude the treatment of spatially extended systems showing high-dimensional chaos. To overcome this, it has been suggested by Grassberger [6] and Mayer-Kress and Kaneko [7] to calculate "dimension densities" for spatially extended systems with the GPA. However, the results have often been criticized. It is argued that inherent systematic errors of the GPA typically lead to an underestimation of the calculated density which might be interpreted erroneously as an indication of deterministic behavior. Torcini *et al.* [8] have carried out very careful investigations of this topic, concluding that it is not possible to distinguish between spatiotemporal chaos and noise relying on the estimate of the dimension density calculated with the GPA or similar techniques.

In this Letter we present numerical results for different model systems showing spatiotemporal chaos. It was not possible to get unequivocal results from an unmodified GPA when applying it to comparatively short time series. We correct these inherent systematic errors by normalizing the results to the correlation sums calculated analytically for an equivalent random attractor. With this modification we show that on the basis of the GPA, one cannot only distinguish between noise and spatiotemporal chaos, but one is also enabled to obtain good quantitative estimates of the dimension density. In addition, we attain a quantification of the space-time coupling which allows us to distinguish between systems consisting of many regions of uncoupled temporal chaos and systems consisting of coupled regions producing real spatiotemporal chaos.

The Letter is structured in the following way. First we will describe the characteristics of the GPA followed by the introduction of our modification to the interpretation of the results. After this we will briefly review the definition and the practical calculation of dimension densities using Lyapunov spectra. In the last part the dimension densities of spatiotemporal chaotic systems, calculated with the modified GPA, will be compared with the results obtained from the Lyapunov spectra using the equations of motion.

In the Grassberger-Procaccia analysis, the attractor is represented in an m-dimensional space, thus the considered time series consists of vectors $\mathbf{x}(t) = (x_1(t), x_2(t), ..., x_m(t))$. The components $x_i(t)$ represent different dynamical variables of the system [9]. Afterwards, for a set of N_p reference points, we calculate the probability to find a second point in a distance less than a given radius r by evaluating the correlation sums $C(r,m) = \lim_{N_p\to\infty}(1/N_p^2)\sum_{i\neq j}\Theta(r - |\mathbf{x}(t_j) - \mathbf{x}(t_i)|)$. The correlation dimension D_2 can now be determined from the slope:

$$D_2 = \lim_{r\to 0}\lim_{m\to\infty}\frac{d\log C(r,m)}{d\log r}. \tag{1}$$

In the practical application we have to deal with finite numbers of vectors and with a finite noise level. For that we have to evaluate the slope (1) at finite length scales. In principle the slope can be calculated even for large values of r requiring less data points, but the estimation of D_2 is believed to be reliable only if the same dimension

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is calculated in a sufficiently large interval of $\log(r)$. In the regions where r becomes comparable with the size of the attractor we encounter an underestimation of D_2 [10] that depends systematically on r. This is due to the fact that for reference points closer to the edge of the attractor than r, the probabilities are smaller than expected for dimension D_2 [see inset of Fig. 1(b)].

We choose the following way for dealing with the errors resulting from the influence of the overall shape of the attractor. We leave the calculated correlation sum C unchanged and express the necessary changes in terms of a corrected radius $r_{\text{eff}}(r)$. Since empty regions outside of the attractor are considered within r, this effective radius r_{eff} will be smaller than the original r. The calculation of r_{eff} is done by considering the correlation sum $\tilde{C}(r,m)$ of a random attractor of the same shape. Knowing that the slope $\Delta \log \tilde{C}(r,m)/\Delta \log r_{\text{eff}}$ for the random attractor has to be m, the explicit calculation of $\Delta \log r_{\text{eff}}$ can be replaced by $\Delta \log \tilde{C}(r,m)/m$, so we are ending up with

$$D_2/m = \frac{\Delta \log C(r,m)}{\Delta \log \tilde{C}(r,m)}. \tag{2}$$

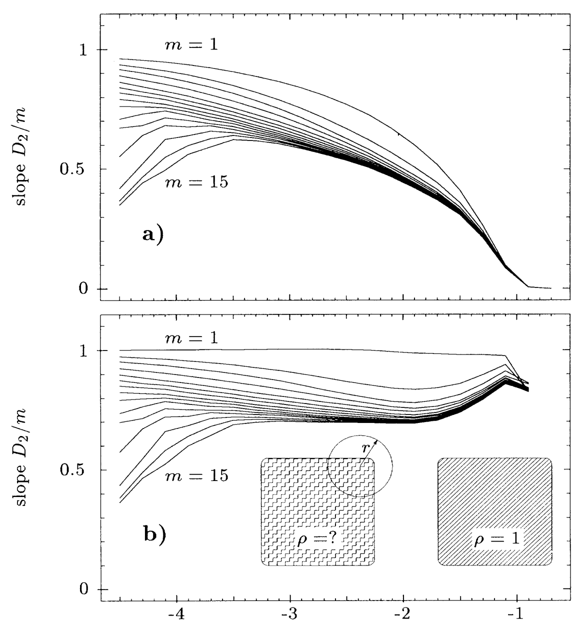

FIG. 1. The dimension density obtained by the Grassberger-Procaccia analysis of a spatiotemporal chaotic time series of $N_p = 10^5$ vectors is plotted. (a) Using standard GPA, the calculated slope depends on the considered radius r; no unequivocal value for the dimension density can be obtained. (b) Normalizing the correlation sums on the results of a random signal according to (2) and (3) gives a good convergence of the dimension density ρ_2 in the whole interval between the radius where more than half of the attractor is covered ($\log_2 r \approx -0.8$) and the region where the number of data points is too small to give reliable statistics for higher dimensions ($\log_2 r \approx -3.5$). The inset illustrates how a large radius r exceeds the area of the attractor, making it neccessary to introduce a smaller effective radius r_{eff}. As described in the text, this effective radius is calculated by considering a random attractor ($\rho = 1$) of the same shape.

The correlation sum $\tilde{C}$ is calculated for a random attractor with a probability distribution that is similar to the distribution of the original data on a macroscopic scale, but that has a nonfractal structure on the microscopic scale. For the case $m = 1$, this property can be achieved by dividing the range of the quantities in our time series into a limited number of classes N_c and assuming constant probability density within each class. The details of this partitioning are not crucial as long as the global shape of the 1-dimension probability distribution can be reproduced; in our simulations we are using $N_c = 42$. We denote the half-width of these classes by a and the probability density in class j will be called p_j. For such a distribution the correlation integral can be calculated analytically using directly the definition of $C(r,m)$. Every reference point has to be taken into account according to the probability of its class p_n. For this class there will be a contribution by the classes that are located within the radius r. These are at least the classes in the interval $n \pm h$ with the integer $h = \text{int}(r/2a)$ and an additional contribution of the classes at the boundaries, depending on the distance of the boundaries $d = r - 2ha - a$. The summation over all intervals results in the following expression for the correlation sum for random numbers [11]:

$$\tilde{C}(r,1) = \sum_{n=1}^{N_c} p_n \left[4a^2 \sum_{j=n-h}^{n+h} p_j - \frac{p_{n+h} + p_{n-h}}{2}(a-d)^2 + \frac{p_{n+h+1} + p_{n-h-1}}{2}(a+d)^2 \right]. \tag{3}$$

If maximum norm is used to calculate the correlation sums for arbitrary embedding dimension m and the multidimensional probability distribution factorizes like $p_{i(x)j(y)} = p_{i(x)}p_{j(y)}$, the corresponding sum for random numbers can be obtained as the product of all the $\tilde{C}(r,1)$ in every direction. If in addition the probability p_j is the same in all directions (which is normally true for the spatiotemporal systems we will analyze below), we get the simple expression $\tilde{C}(r,m) = \tilde{C}(r,1)^m$.

Let us now consider the systems showing spatiotemporal chaos. As standard models for spatially extended systems we are using coupled map lattices (CML) [5,12] consisting of logistic maps or tent maps as well as chaotic neural networks [13]. Both of them are discrete in space and time and are defined by a nonlinear local mapping and a spatial coupling. The definitions of the systems are briefly described in the footnotes [14,15]. When the equations of motion are known, the spectrum of Lyapunov exponents can be calculated [16] even for a high-dimensional system to characterize the dynamical properties of the attractor [17,18]. It turns out that for a sufficient size N of the system the distribution of the N Lyapunov exponents λ_i can be described by a unique function $L(i/N) = \lambda_i$ [6,19]. The Lyapunov exponents are connected to the fractal dimension of the attractor via the Kaplan-Yorke formula [1,20], allowing us to calculate a Lyapunov dimension D_L directly from the spectrum. This leads to the conclusion that the dimension of such a system is an "extensive" quantity that grows linearly with the size of the system. For that, it is reasonable to define a size independent "specific" dimension or dimension density $\rho_L := \lim_{N\to\infty} D_L/N$.

We calculate this quantity for many parameter values of the considered systems and find a very good convergence of ρ_L in dependence of N. Therefore the attractor's dimension of a large system can be determined from the properties of a smaller system. If the boundary conditions can be neglected, the considered smaller system may also be a subsystem of the larger system. We have checked this idea by calculating the Lyapunov dimension D_L in a chaotic network, considering only subsystems of size N_s while the influence of the rest of the system is neglected. It turns out that in this case the dimension D_L is growing linearly with the size of the subsystem N_s and the slope is consistent with the dimension density ρ_L calculated for larger systems.

These results imply that it might also be possible to use the GPA for the calculation of a quantity

$$\rho_2 = D_2/m \tag{4}$$

by choosing an equivalent procedure to analyze measured time series. One will determine the dimension density of a large system by reconstructing the attractor from the time series of a small subsystem and calculating the dimension D_2 with the GPA. In contrast to the standard GPA we do not expect a convergence but a linear growth of the calculated dimension with increasing size of the subsystem $N_s = m$.

In order to test this assumption we try to calculate the dimension density ρ_2 for a system of 1000 coupled tent maps ($a = 1.7, \epsilon = 0.2$) [14] using a time series of $N_p = 10^5$ vectors $\mathbf{x}(t) = (x_t^1, x_t^2, ..., x_t^m)$. The results of the original GPA as shown in Fig. 1(a) are not so easy to interpret since a different value of ρ_2 is calculated for every radius r. There might be a convergence for smaller values of r, but the number of 10^5 points is too small to achieve any reliable statistics in this region. The fact that a set of completely different values for ρ_2 can be concluded from this result and one cannot exclude that $\rho_2(r)$ will reach 1 for smaller values of r was the main point of criticism on the application of GPA on high-dimensional systems. However, when we are normalizing the results of the GPA in order to avoid the systematic errors in the way described above according to (2) and (3) we end up with a satisfying convergence of the dimension density in a comparable large range of r [Fig. 1(b)]. Thus it is possible to determine ρ_2 with an accuracy better than 10%.

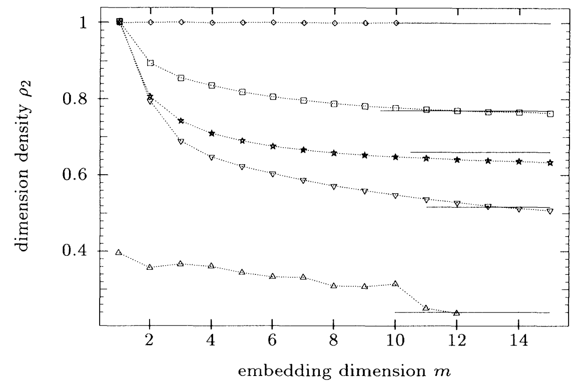

FIG. 2. The convergence of the dimension density in dependence of the considered size of the subsystem m is plotted (symbols) and compared with the results calculated from the analysis of the Lyapunov dimension (solid lines). A good quantitative agreement can be observed. The results for the following systems are plotted (compare [14,15]): $\diamond$, Gaussian random numbers; $\square$, CML with logistic map ($a = 4, \epsilon = 0.4$); $\star$, CML with tent map ($a = 2, \epsilon = 2/3$); $\triangledown$, CML with logistic map ($a = 3.9, \epsilon = 0.8$); $\triangle$, neural network ($k = 30, g = 1$).

Applying this method to time series of different systems showing spatiotemporal chaos as well as to random numbers we observe in any case a convergence of the calculated dimension density as depicted in Fig. 2. The dimension densities for all deterministic systems are well below the value of $\rho_2 = 1$ calculated for the random sequence. Furthermore, we find a good convergence of ρ_2 towards the values of ρ_L calculated directly from the Lyapunov spectra via the Kaplan-Yorke formula, indicated by the solid lines in Fig. 2 [21]. The dimension densities for many different systems are summarized in Table I. It shows that normalized GPA enables us to calculate the densities ρ_2 from time series quite correctly. It should be emphasized that the total dimension cannot be cal-

TABLE I. The dimension densities ρ_2 calculated using the normalized GPA are compared with the ρ_L calculated directly from the Lyapunov dimension. Both methods are applied to different systems with different sets of parameters [14,15] showing that the results are consistent.

System	Parameter	ρ_L	ρ_2
Random		1.0	1.0
Tent[a]	$a = 1.7$, ϵ=0.4	0.81	0.70
Logist	$a = 3.9$, ϵ=0.8	0.51	0.51
Tent	$a = 2.0$, ϵ=0.66	0.66	0.63
Logist	$a = 4.0$, ϵ=0.5	0.65	0.67
Logist	$a = 4.0$, ϵ=0.4	0.76	0.77
Neural	$k = 30$, g=1	0.24	0.24

[a]GPA of the tent-map system using $N_p = 10^6$ vectors indicates that ρ_2 is really smaller than ρ_L in this case.

TABLE II. The propagation velocities v of defects that are calculated directly from the simulations are compared with the quotient of the "spatial" and the "temporal" dimensions σ/ρ. Since an error of approximately 10% can be assumed for the calculated dimensions, the results are in good agreement.

System	Parameter	v	σ/ρ
Tent	$a = 1.7$, ϵ=0.4	0.50	0.54
Tent	$a = 2.0$, ϵ=0.66	0.78	0.7
Logist	$a = 3.9$, ϵ=0.8	0.68	0.58
Logist	$a = 4$, ϵ=0.4	0.51	0.55

culated directly with any realistic amount of data and computer time, since a density of $\rho_2 = 0.8$ means an attractor dimension of $D_2 = 800$ in our system of 1000 coupled maps.

Following an idea of Grassberger [6] we characterize the coupling between the spatial and the temporal degree of freedom by the average propagation velocity (or "speed of light") v of defects [22]. In a system which is not coupled spatially, defects would not propagate at all (i.e., $v = 0$). For finite spatial coupling we generally expect a velocity that increases with the coupling strength. In simulations, an undisturbed and a disturbed system can be iterated separately and the velocity v can be determined directly from the difference of both results. If solely the time series of a large system are available, v can be calculated from the dimension density of a "space series" of vectors formed from subsequent maps in spatial direction according to $\mathbf{x}(i) = (x_t^i, x_{t+1}^i, ..., x_{t+m-1}^i)$. Using these vectors to represent the attractor and applying our method we obtain a dimension density σ that is connected with the propagation velocity via $v = \sigma/\rho$ [6]. In Table II we compare the values calculated directly from the simulations to the values obtained from the analysis of the time series. Again we find a good quantitative agreement of both results.

In conclusion, we have shown that time series of systems showing high-dimensional spatiotemporal chaos can be analyzed successfully by calculating the dimension density with a normalized Grassberger-Procaccia algorithm. The good quantitative agreement with values which are calculated directly from the equations of motion makes us believe that this is a reliable method even for systems for which these equations are not known. Since the normalization of the correlation sums on the results for random numbers enables us to choose comparable large radii r, this method should work even for signals with small additional noise.

* Electronic address: bauer@physik.uni-frankfurt.de

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