Solution of Some Integrable One-Dimensional Quantum Systems

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We investigate a family of one-dimensional multicomponent quantum many-body systems with an exchange interaction based on the inverse square potential. We show these systems to be integrable, and exploit this integrability to completely determine the spectrum, degeneracy, and thermodynamics. The periodic inverse square case is worked out explicitly. Next, we show that in the limit of strong interaction the "spin" degrees of freedom decouple and thus we obtain a complete solution to a lattice system introduced recently by Shastry and by Haldane. Finally, we emphasize the simple explanation for the high multiplicities found in this model.

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The plan of this paper is to first review integrable systems and the Lax technique, show that certain modified systems with an exchange interaction are integrable, give the general solution to these systems and an explicit solution for an important example, then to produce from this solution by an appropriate limit the solution to a family of lattice problems, and finally we discuss what it all means. The impatient reader may skip to the closing paragraphs for a summary of the results.

If we consider a classical system of N one-dimensional particles, then it has been shown by Lax [1], Moser [2], and Calogero [3] that for certain potentials one can find two Hermitian $N \times N$ matrices L and A that obey the Lax equation $dL/dt = i[A, L]$. Thus L evolves by a unitary transformation generated by A, and hence $\det[L - \omega I]$ is a constant of motion. Expanding the determinant in powers of ω, we find N integrals of motion $\det[L - \omega I] = \sum_{0 \leq j \leq N} J_j(-\omega)^{N-j}$, $j = 1, \ldots, N$. Further, these integrals have been shown to be in involution, and thus the system is integrable.

The two matrices are given as $A_{jk} = \delta_{jk} \sum_{l(\neq j)} \gamma(x_j - x_l) + (1 - \delta_{jk})\beta(x_j - x_k)$, and $L_{jk} = \delta_{jk}p_j + i(1 - \delta_{jk})\alpha(x_j - x_k)$. The functions $v(x)$, $\gamma(x)$, and $\beta(x)$ are even, while $\alpha(x)$ is odd. They obey the equations, $v'(x) = -2\alpha(x)\beta(x)$, $\beta(x) = -\alpha'(x)$, and $\alpha(x+y)[\gamma(y) - \gamma(x)] = \alpha'(y)\alpha(x) - \alpha'(x)\alpha(y)$. Calogero [3] has shown that the most general solution to these equations is given in terms of the Jacobi elliptic function $\mathrm{sn}(x|m)$ as

$$\alpha(x|m, \lambda, \kappa) = \lambda\alpha(\kappa x) = \frac{\lambda}{\mathrm{sn}(\kappa x|m)} ,$$

$$\gamma(x|m, \lambda, \kappa, c) = \lambda\kappa\gamma(\kappa x) + c = \lambda\kappa\alpha^2(\kappa x) + c = \frac{\lambda\kappa}{\mathrm{sn}^2(\kappa x|m)} + c ,$$

$$v(x) = \lambda^2\alpha^2(\kappa x) + v_0 .$$

Calogero [4] has also demonstrated that if one replaces the classical dynamical variables with the corresponding quantum mechanical operators, $\det[L - \omega I]$ is well defined with no ordering ambiguity, and the quantum mechanical commutator $[\![H, \det[L - \omega I]]\!] = 0$. Thus, the J_j are still constants of motion. Finally, Calogero showed that $[\![\det[L - \omega I], \det[L - \omega' I]]\!] = 0$, and thus the quantum system is also completely integrable.

For the general classical system integrability tells us something concrete, namely that the motion in terms of action-angle variables is on a torus. However, for the general quantum system integrability seems to buy one almost nothing. The exception is for those special cases which support scattering, i.e., systems which fly apart when the walls of the box are removed. In these cases, in the distant past and future the Lax matrix L approaches a diagonal matrix, so that $\det[L - \omega I] = \prod_{1 \leq j \leq N}(p_j - \omega)$. Thus the individual momenta p_j are conserved in a collision, and hence the wave function is given asymptotically by Bethe ansatz. Sutherland [5] has exploited this fact to completely determine, in the thermodynamic limit, properties of systems interacting by potentials of the forms $v(x) = g/x^2$, $g/\sin^2(x)$, and $g/\sinh^2(x)$, including the Toda lattice. We emphasize that no features of the proof of integrability are needed. All we need is to know it to be integrable, by whatever method. (We might even just assume it to be integrable, and deduce the consequences.)

Although the previous problems are multicomponent problems, in fact statistics enters only trivially, due to the strong repulsion of the potential at the origin. Polychronakos [6] has modified this problem, allowing particles to penetrate by means of an exchange interaction. He then proves the integrability of this modified problem. We now offer an alternative proof of integrability. If P_{jk} is the permutation operator that permutes particles j and k, let us modify the Lax matrices to be $A_{jk} = \delta_{jk} \sum_{l(\neq j)} P_{jl}\gamma(x_j - x_l) + (1 - \delta_{jk})P_{jk}\beta(x_j - x_k)$,

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and $L_{jk} = \delta_{jk}p_j + i(1-\delta_{jk})P_{jk}\alpha(x_j - x_k)$. We assume that the new potential v_{jk} commutes with P_{jk}. We then seek to satisfy a quantum Lax equation of the form$[\![H,L]\!] = [L,A]$. Here the first fancy commutator is a quantum mechanical commutator between operators, while the second commutator is an ordered matrix commutator. In order that these equations be satisfied, one finds that the functions $\alpha(x)$, $\beta(x)$, and $\gamma(x)$ must satsify exactly the same equations as before. The potential now is

$$v_{jk}(x) = \lambda^2\alpha^2(\kappa x) + \lambda\kappa\gamma(\kappa x)P_{jk} = \frac{\lambda^2 + \lambda\kappa P_{jk}}{\mathrm{sn}^2(\kappa x|m)} .$$

We will henceforth be interested in a particular case. Let $\kappa = i$, $\lambda = i$, $c = -1$, and $m \to 1$, so that $\alpha(x) \to \cot(x)$, $\gamma(x) \to -1/\sin^2(x) = -1 - \alpha^2(x)$, and $\alpha'(x) = -\beta(x) \to -1/\sin^2(x) = \gamma(x)$. Restoring scale factors, we have a potential $v_{jk}(x) = (\lambda^2 - \lambda P_{jk})\kappa^2/\sinh^2(\kappa x)$. (We could as well have a sine instead of a sinh.)

The significance of this case is for the form of the Lax A matrix. Since $\beta(x) = -\gamma(x)$, then defining a vector η with $\eta_j = 1$, we see $A\eta = \eta + A = 0$. This allows us to construct constants of motion by $I_n = \eta^\dagger L^n \eta$, since

$$\begin{aligned}[\![H,I_n]\!] &= \eta^\dagger[\![H,L^n]\!]\eta = \eta^\dagger \sum_{0<j<N-1}\{L^j[\![H,L]\!]L^{N-1-j}\}\eta \\ &= \eta^\dagger \sum_{0<j<N-1}\{L^j[A,L]L^{N-1-j}\}\eta \\ &= \eta^\dagger\{AL^{N-1} - L^{N-1}A\}\eta = 0 .\end{aligned}$$

By Jacobi's relation for commutators, $[\![I_n,I_m]\!]$ is a constant of motion, and since this is a system that supports scattering, we see $[\![I_n,I_m]\!] \to 0$, and hence the system is completely integrable. Thus once again, the individual momenta p_j are conserved in a collision, and hence the wave function is given asymptotically by Bethe ansatz. This allows one to completely determine, in the thermodynamic limit, properties of systems interacting by potentials $v(x) = (\lambda^2 - \lambda P)/x^2$ and $(\lambda^2 - \lambda P)/\sinh^2(x)$, as well as the periodic versions of these such as $(\lambda^2 - \lambda P)\pi^2/L^2\sin^2(\pi x/L)$ in the limit as $L \to \infty$, density finite.

The input for the Bethe ansatz wave function is the two-body scattering matrix, since the N-body scattering is simply a product of all $N(N-1)/2$ scatterings. At the same time, we can gain some confidence by confirming the consistency of the two-body scattering matrix; it must satisfy the Yang-Baxter equations. With the relative coordinate $r = x_2 - x_1$, the potential is $v(r) = (\lambda^2 - \lambda P)/r^2$, so depending on whether the wave function is even or odd, the potential is $v_\pm(r) = \lambda(\lambda \mp 1)/r^2$. The radial equation then is $-\varphi'' + \sigma(\sigma-1)\varphi/r^2 = k^2\varphi$, with $\sigma = \lambda$ or $\lambda + 1$ according to whether the wave function is even or odd. The solution is given as $\varphi(r) = \sqrt{r}J_{\sigma-1/2}(kr)$, $r > 0$. For the even and odd solutions then we have $\varphi_\pm(r) \to e^{-ikr} \pm e^{-i\pi\lambda}e^{ikr}$, $r \to \infty$. To construct a scattering state in which particle 1 with momentum k_1 is incident on particle 2 with momentum $k_2, k_1 > k_2$ we take the linear combination

$$\begin{aligned}\frac{1}{\sqrt{2}}[\phi_+ + \phi_-] &\to e^{i(k_1x_1 + k_2x_2)}, \quad x_1 \ll x_2 , \\ &\to e^{-i\pi\lambda}e^{i(k_1x_1+k_2x_2)}, \quad x_2 \ll x_1 .\end{aligned}$$

Thus, there is no reflection, and the particles pass through each other with only a phase shift $e^{-i\pi\lambda}$. Consistency is not an issue. This is a surprising result with important consequences.

We now place the particles on a large ring of circumference L, making the problem periodic. We thus need to replace the potential with some periodic potential, such as $v(x) = (\lambda^2 - \lambda P)\pi^2/L^2\sin^2(\pi x/L)$, which reduces to $(\lambda^2 - \lambda P)/x^2$ in the limit $L \to \infty$, and which therefore has the same thermodynamic limit. To repeat: We will be giving the exact solution for the inverse sine squared exchange potential in the thermodynamic limit of an infinite system with finite density.

Taking the jth particle around the ring of circumference L, it scatters from every other particle with a net phase shift that must be unity, to insure that the wave function is periodic. Thus the k's satisfy the equation $\exp[ik_jL - i\pi\lambda\sum_{l(\neq j)}\mathrm{sgn}(k_j - k_l)] = 1$, or upon taking the logarithm, $k_j = k_j^0 + \pi\lambda/L\sum_{l(\neq j)}\mathrm{sgn}(k_j - k_l)$. Here $k_j^0 = 2\pi$ integer$/L$ are the noninteracting momenta with $k_1^0 \geq k_2^0 \geq \cdots \geq k_N^0$. We easily solve for the k's obtaining $k_j = k_j^0 + \pi\lambda(N + 1 - 2j)/L$. Then the energy is

$$E = \frac{1}{2}\sum_{1\leq j\leq N}k_j^2 = \frac{1}{2}\sum_{1\leq j\leq N}(k_j^0)^2 + \frac{\pi\lambda}{L}\sum_{1\leq j\leq N}k_j^0(N+1-2j) + \pi^2\lambda^2N(N^2-1)/6L^2 ,$$

while the momentum is $P = \sum_{1\leq j\leq N}k_j = \sum_{1\leq j\leq N}k_j^0$. All of this is just a repetition of the old results [5] for the usual $1/r^2$ potential, which in fact is a special case of this interaction with a single species of particle.

Although the energy eigenvalues are exactly the same, these levels are now highly degenerate. In fact, the degeneracy is exactly the same as the free particle degeneracy when $\lambda \to 0$. Thus, let us introduce occupation numbers $\nu_\alpha(k)$, equal to the number of particles of species α with noninteracting momenta k. Thus $\nu_\alpha(k) = 0, 1$ for fermions, and $\nu_\alpha(k) = 0, 1, 2, \ldots$ for bosons. Then the energy E is given by

$$\begin{aligned}E/L &= \frac{1}{4\pi}\sum_\alpha\int_{-\infty}^{\infty}dk\,\nu_\alpha(k)k^2 + \frac{\lambda}{8\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}dk\,\nu_\alpha(k)\int_{-\infty}^{\infty}dk'\,\nu_\beta(k')|k-k'| + \pi^2\lambda^2d^3/6 \\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty}dk\,\nu(k)k^2 + \frac{\lambda}{8\pi}\int_{-\infty}^{\infty}dk\,\nu(k)\int_{-\infty}^{\infty}dk'\,\nu(k')|k-k'| + \pi^2\lambda^2d^3/6 .\end{aligned}$$

Here, the density of species α is $d_\alpha = N_\alpha/L = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \nu_\alpha(k)$, $\nu(k) = \sum_\alpha \nu_\alpha(k)$, and $d = N/L = \sum_\alpha d_\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \nu(k)$ is the total density. The degeneracies are now expressed in terms of the usual entropy S, as $S/L = (1/2\pi)\sum_\alpha \int_{-\infty}^{\infty} dk(\pm 1)(1 \pm \nu_\alpha)\ln(1 \pm \nu_\alpha) - \nu_\alpha \ln(\nu_\alpha)$. (Here, and in what follows, the upper sign is for bosons, while the lower sign is for fermions.)

We can now determine the equilibrium occupation numbers $\nu_\alpha(k)$ as those which maximize $P/T = S/L - E/TL + \sum_\alpha \mu_\alpha N_\alpha/TL$. This gives the familiar expression $\nu_\alpha = 1/[e^{(\varepsilon-\mu_\alpha)/T} \mp 1]$. In this expression, $\varepsilon = k^2/2 + (\lambda/2)\int_{-\infty}^{\infty} dk'\, \nu(k')|k-k'| + \pi^2\lambda^2 d^2/2$. Differentiating, we find for the interacting momenta p that appear in the asymptotic Bethe ansatz, $p = \varepsilon' = k + (\lambda/2)\int_{-\infty}^{\infty} dk' \nu(k')\mathrm{sgn}(k-k')$. Differentiating once again, $p' = \varepsilon'' = 1 + \lambda\nu = p\, dp/d\varepsilon = (d/d\varepsilon)(p^2/2)$. Thus, integrating, we obtain $p^2/2 = \varepsilon + \lambda T \sum_\alpha (\pm 1)\ln(1 \mp e^{(\mu_\alpha - \varepsilon)/T})$. We may then go back and write the densities as $2\pi d_\alpha = \int_{-\infty}^{\infty} dp \frac{\nu_\alpha}{1+\lambda\nu}$. This gives us a complete solution for the thermodynamics.

For instance, if we consider a system consisting of two species of bosons, say + and −, then taking $\mu_\pm = \mu \pm h$, we find $e^{-p^2/2\lambda T} = [e^{(\varepsilon-\mu)/T} + \cosh(h/T)]^2 - \sinh^2(h/T)$, so $e^{(\varepsilon-\mu)/T} = [e^{-p^2/2\lambda T} + \sinh^2(h/T)]^{1/2} - \cosh(h/T)$. This then allows us to write $\nu_\pm$ as an explicit function of p, which may be integrated to give $d_\pm$ as a function of $h_\pm$, and which in turn can be integrated with respect to $h_\pm$ to give the free energy, which can be referenced to the known one-component system at $h_+ \to \infty$.

At this point, let us look at the ground state energy as a function of the densities $\{d_\alpha\}$. This is easily done, and it is only for the ground state when $T = 0$ that we expect singularities. We assume f species of fermions with densities $d_1 \le d_2 \le \cdots \le d_f$, and a total of d_B bosons of any species, so the total density is d. Then the occupation numbers are the free particle ground state occupation numbers, and the integrals in the expression for the energy are easily evaluated to give

$$6E/\pi^2 L = (1+2\lambda)\sum_{1\le\alpha\le f} d_\alpha^3 + 3\lambda d_B \sum_{1\le\alpha\le f} d_\alpha^2 + \lambda \sum_{1\le\alpha<\beta\le f} d_\alpha(d_\alpha^2 + 3d_\beta^2) + \lambda^2 d^3 .$$

Because this expression is only valid for a particular ordering of the densities, the ground state does have singularities whenever $d_\alpha = d_\beta$. A Taylor expansion gives the singularity as $E/L \approx |d_\beta - d_\alpha|^3$, so the susceptibilities do not diverge. The singularity structure, and even the nature of the singularities, is very similar to that found previously for a short-ranged lattice exchange model [7].

Examining the potential energy as $\lambda \to \infty$, we see that the particles crystallize into a lattice with lattice constant $1/d$, and so the elastic modes of this lattice, and the compositional or "spin" modes on the lattice separate. We thus have as $\lambda \to \infty$

$$H \to \tfrac{1}{2}\sum_{1\le j\le N} p_j^2 + \lambda^2 \sum_{1\le j<k\le N} \frac{\pi^2}{L^2\sin^2[\pi(x_j - x_k)/L]} - \lambda d^2 \sum_{1\le j<k\le N} \frac{\pi^2 P_{jk}}{N^2 \sin^2[\pi(j-k)/N]} = H_e + \lambda d^2 H_s .$$

The Hamiltonian for the elastic modes is just the familiar one-component system [5], although with a coupling constant λ^2. We have defined the spin Hamiltonian H_s with exchange constant unity, although the effective exchange constant of the last term of H is λd^2. The two-species problem was first studied independently and simultaneously by Shastry [8] and by Haldane [9]. Thus we see that by an appropriate limiting process we may determine the thermodynamics of this spin Hamiltonian exactly. For the spin problem, the appropriate concentrations are $m_\alpha = N_\alpha/N = d_\alpha/d$.

Let $F(T,\{N_\alpha\},L)$ be the free energy for the original continuum exchange problem. Then it is seen on dimensional grounds to be of the form $LT^{3/2}f(\{d_\alpha\}/T^{1/2})$. Given a set of "chemical potentials" $\{h_\alpha\}$, such that $\sum_\alpha h_\alpha = 0$, then we make a partial Legendre transformation,

$$F(T,\{N_\alpha\},L) - \sum_\alpha h_\alpha N_\alpha = F(T,\{h_\alpha\},N,L) = LT^{3/2} f(d/T^{1/2},\{h_\alpha\}/T^{1/2}) .$$

(Free energies are identified by their variables.) Then $N_\alpha = -\partial F/\partial h_\alpha$, so if $d_\alpha(d,\{h_\alpha\}) = -\partial f(d,\{h_\alpha\})/\partial h_\alpha$, we can write $d_\alpha = T^{1/2} d_\alpha(d/T^{1/2},\{h_\alpha\}/T)$, which for $T=1$ is simply d_α.

Returning now to the limit as $\lambda \to \infty$, and setting $T = 1$, we have $f(\{d_\alpha\}) \to f_1(d) + d f_s(1/\lambda d^2,\{m_\alpha\})$. Here $f_1(d)$ is the free energy density for the one-component system [5] with coupling constant λ^2, and $f_s(T,\{m_\alpha\})$ is the free energy per site for the spin system. Again making the partial Legendre transformation, we find $f_s(T,\{h_\alpha\}) = \lim_{\lambda\to\infty}\sqrt{\lambda T}[f(1/\sqrt{\lambda T},\{h_\alpha\}/T) - f_1(1/\sqrt{\lambda T})]$. Remember, in the functions on the right-hand side, the first argument represents the density $d = 1/\sqrt{\lambda T} \to 0$ as $\lambda \to \infty$. Finally, the relationship is a little clearer in terms of the concentrations, $m_\alpha = \lim_{\lambda\to\infty} d_\alpha/d(\{\mu_\alpha\})$, where the chemical potentials are $\{\mu_\alpha\} = \{\mu + h_\alpha/T\}$, and μ is adjusted to give $1 = T\lambda d^2$.

Referring to our previous explicit results, in the limit as $\lambda \to \infty$, with $1 = T\lambda d^2$, then $m_\alpha = d_\alpha/d = 1/2\pi = \int_{-\pi}^{\pi} dp\, \nu_\alpha/\nu$, with $\nu_\alpha = 1/[\exp(\varepsilon - h_\alpha/T) \mp 1]$, and $(p^2-\pi^2)/2 = T\sum_\alpha(\pm 1)\ln[1 \mp \exp(h_\alpha/T - \varepsilon)]$. This gives us a complete solution, since the concentrations can be integrated with respect to $\{h_\alpha\}$ to give the free energy. The free energy is referenced with respect to the trivial case of $m_\alpha = 1$ for some α. For a two species problem of either both fermions or both bosons, our results reproduce Haldane's conjecture based on his numerical investigation [9]. We emphasize that one changes the sign of the spin Hamiltonian from ferromagnetic to antiferromagnetic simply by changing the statistics of the particles from Bose to Fermi.

Let us now summarize what we have accomplished. We have investigated a family of one-dimensional multicomponent quantum many-body systems, interacting by an exchange interaction based on the familiar family of integrable systems which includes the inverse square potential. These systems are shown to be integrable. Since these systems support scattering, the asymptotic wave function is given by Bethe ansatz, and consequently the spectrum can be completely determined, in the thermodynamic limit. The interactions that can be solved in this way include $v(x) = (\lambda^2 - \lambda P)/x^2$ and $(\lambda^2 - \lambda P)/\sinh^2(x)$ at finite density, as well as the periodic versions of these such as $(\lambda^2 - \lambda P)\pi^2/L^2 \sin^2(\pi x/L)$ in the limit as $L \to \infty$, density finite.

We explicitly work out the case of the inverse square or inverse sine square potentials, and demonstrate that for this exchange potential particles scatter without reflection. They pass through one another with only a phase shift, and this phase shift is independent of the identities of the particles (and the momentum as well). As a consequence of this perfect transmission, there can be no Bragg reflection to split the energy of the even and odd solutions when the particles pass around the ring. Instead we have degenerate states, and so we could take a linear combination which, for a two-component system, would have all particles of one type going clockwise around the ring, while all particles of the other type go counterclockwise. Yet this state would be an eigenstate of energy. This then explains the very high degeneracy observed in the spectrum; it is just the degeneracy of the free particle system. We can offer other examples of systems without reflection, with similar behavior. In fact, it is clear from the discussion, that by using adiabatic continuity, the quantum numbers of the states can be taken as the corresponding quantum numbers of the free particle states. Singularities occur only in the ground state, and these are exhibited.

Finally, it is demonstrated that for each of these systems, taking the limit of infinite coupling constant appropriately gives the solution to a corresponding lattice problem. One could easily prove the integrability of these lattice systems directly using the Lax technique. However, application of the asymptotic Bethe ansatz directly to the lattice is plagued by umklapp problems; the momenta are pushed by the interaction outside the Brillouin zone [10]. The limiting trick avoids these complications. Taking this limit in our explicit example, we obtain a complete solution to a corresponding lattice system introduced recently by Haldane and by Shastry. Our solution reproduces Haldane's numerical results. This system exhibits very high multiplicities; we find these multiplicities to be just the familiar degeneracy of free particles. It arises naturally from the absence of reflection, and hence of Bragg reflection and the consequent splitting of levels, and is nothing more. This seems to us a good explanation.

The success of the asymptotic Bethe ansatz has been rather dramatic, so let us here reiterate the philosophy behind the method. To justify using the asymptotic Bethe ansatz, one needs all of the following: a system which supports scattering, a proof of integrability with integrals that depend only on the asymptotic momenta, and a valid virial expansion. Given these, and using only the two-body scattering data as input, one can deduce all thermodynamic properties, including ground state properties and low-lying excitations, within the phase for which the virial expansion is valid. Often other phases may be reached by symmetry.

We conclude by reemphasizing that there are many other interesting systems to be explored in detail in the future.

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