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New Family of Unitary Random Matrices

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We consider an exactly solvable random matrix model related to the random transfer matrix model for disordered conductors. In the conventional random matrix models the spacing distribution of nearest neighbor eigenvalues, when expressed in units of average spacing, has a universal behavior known generally as the Wigner distribution. In contrast, our model has a single parameter, as a function of which the spacing distribution crosses over from a Wigner to a distribution which is increasingly more Poisson-like, a feature common to a wide variety of physical systems including disorder and chaos.

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Random matrices have been used to describe a variety of physical systems including complex nuclei [1], chaotic systems [2], and disordered mesoscopic conductors [3]. The connection among these very different systems is not clear, but a random matrix approach in each case is based on the assumption that a sufficiently complicated system is better described by a matrix which is as random as possible, subject to certain symmetry requirements (like the Hermiticity of a Hamiltonian or the unitarity of a scattering matrix). The local statistical properties of the characteristic levels of the system are then similar to certain well-known universal properties of the distribution of eigenvalues obtained from the theory of random matrices [4] developed initially by Wigner, Dyson, Mehta, and others for energy levels of complex nuclei. Although there are a variety of possible random matrix ensembles, they all belong to one of three different classes depending on symmetry (orthogonal, unitary, and symplectic), and all of them have some common universal features. Most well known is the result that the probability distribution of the spacing between nearest neighbor eigenvalues in the bulk of the spectrum, expressed in terms of the local average spacing, follows a universal curve [4] known generally as the Wigner distribution. The actual power law with which the distribution goes to zero for zero spacing (showing "level repulsion" characteristic of such systems) and the asymptotics [5] depend on the symmetry and are therefore different for the three different classes of random matrices, but all of them have the same qualitative shape, and are independent of any parameter of the system, once properly scaled.

This means that all systems describable by these random matrices will always have the universal properties associated with such matrices. For example, for disordered mesoscopic conductors, the universality of the spacing distribution manifests itself as the universal conductance fluctuation known to exist in these systems in the metallic regime [6]. On the other hand, it is well known that for sufficiently strong disorder, there is a transition from metal to insulator, and the conductance fluctuation in the insulating regime is expected to be very different from the universal behavior in the metallic regime [7,8]. Similarly, a chaotic (nonintegrable) system exhibiting level repulsion may undergo a transition to a nonchaotic (integrable) regime as a function of some parameter, and the nonchaotic regime does not show level repulsion characteristic of the universal Wigner distribution [9]. It is therefore of great interest to see if any physically relevant as well as analytically tractable random matrix model can incorporate deviations from the universal Wigner distribution. In the present work we will consider a one-parameter model which is qualitatively similar to the transfer matrix models proposed for disordered conductors, and is exactly solvable. We will show that the model indeed shows a transition from a highly correlated Wigner to an uncorrelated Poisson-like level spacing dis-

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tribution as a function of the parameter.

To illustrate the general framework, consider a physical system described by an $N\times N$ matrix X with eigenvalues x_i $(a\le x_i\le b,\ i=0,\ldots,N-1)$. Within a maximum entropy ansatz [3], the ensemble of all random X matrices consistent with given symmetries (Hermiticity, time reversal, etc.) subject to some physical constraint (e.g., given average density of eigenvalues) has a distribution of eigenvalues that can be written quite generally in the form [3]

$$P\{x_0,\ldots,x_{N-1}\}=\prod_{m<n,0}^{N-1}|x_m-x_n|^{\alpha}\prod_{k=0}^{N-1}\exp[-V(x_k)]\,. \tag{1}$$

Here α is a symmetry parameter and is equal to 1, 2, or 4 for orthogonal, unitary, and symplectic symmetries [4], respectively. For simplicity we will consider $\alpha=2$ only, which corresponds to the case where time reversal symmetry is broken (e.g., due to the presence of a magnetic field). One can define an effective "Hamiltonian" H of the eigenvalues by $P=\exp(-\alpha H)$, consisting of a two-particle "interaction" term which is logarithmically repulsive, and a single-particle "potential" term $V(x)$, which serves to confine the eigenvalues. While the two-particle interaction term is determined by symmetry considerations alone, and is independent of any model parameter, different models within the same symmetry classification correspond to choosing different forms for the single-particle potential V (keeping the interacting part the same) which can be thought of as a Lagrange multiplier function that fixes, e.g., the eigenvalue density (corresponding, e.g., to a given mean value of conductance in the case of a disordered system). In general, given an explicit form for V, it is possible in principle to calculate any n-point correlation function for this distribution exactly for any given N, based on the use of orthogonal polynomials [4]. One defines a family of orthogonal polynomials $P_n(x)$, with given weight $V(x)$ such that

$$\int_a^b dx\, e^{-V(x)}P_n(x)P_m(x)=\delta_{n,m}h_n\,. \tag{2}$$

A two-point kernel $K_N(x,y)$ is defined as

$$K_N(x,y)=e^{-[V(x)+V(y)]/2}\sum_{n=0}^{N-1}\frac{1}{h_n}P_n(x)P_n(y)\,. \tag{3}$$

The eigenvalue density and the spacing distribution can then be calculated in terms of this kernel. For translationally invariant kernels the spacing t between nearest eigenvalues, in units of average local spacing, for large N, has the spacing distribution $p(t)$ given by [4]

$$p(t)=\frac{d^2E(t)}{dt^2},\quad E(t)=\det(1-\bar{K}_t)\,, \tag{4}$$

where $\bar{K}_t$ is the kernel divided by the local density and restricted to an interval of length t.

So far all the random matrix models investigated analytically in detail have led to the same universal results, and have employed choices for the potential such that the associated polynomials are one of the classical orthogonal polynomials. On the other hand, all random transfer matrix models proposed for disordered conductors that can describe both the metallic and the insulating regions [3,10,11] require a potential $V(x)$ which behaves as $(\ln x)^2$ for large x and cannot be associated with any classical orthogonal polynomial. For example, Ref. [11] has extracted the functional form of the potential [for the matrix $X=TT^{\dagger}+(TT^{\dagger})^{-1}-2I$, where T is the transfer matrix characterizing the disordered conductor and I is the unit matrix; the conductance g is given in terms of the eigenvalues of X by $g=\sum_i^N 1/(1+x_i)$] by considering numerical diagonalization of the transfer matrix obtained either directly from a microscopic tight binding Anderson Hamiltonian with random site energies or from a network of disordered quantum wires, for various values of disorder. The discrete points in Fig. 1 reproduce the $V(x)$ from Ref. [11] which, as noticed in that reference, can be fitted very well with a function of the form $a[\ln(1+bx)]^2$, where a and b are constants depending on disorder, etc. The two-point correlation functions obtained from these potentials fit very well [11] the two-point functions in the bulk directly obtained numerically from the microscopic model, showing that such a form for the potential can be taken as a qualitatively correct phenomenological model for disordered conductors. Note that this potential has the same qualitative behavior as in the model discussed in detail in Ref. [3], where the single-particle

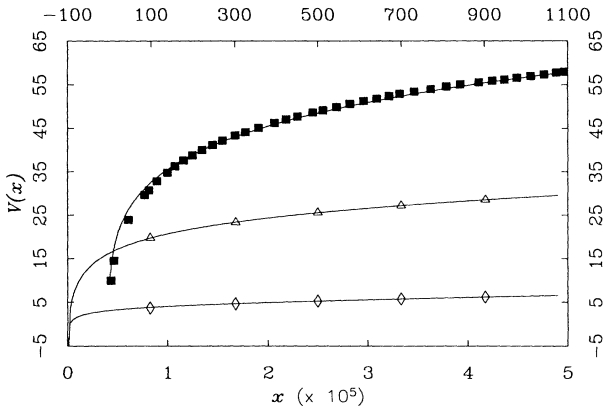

FIG. 1. The single-particle potential $V(x)$. The discrete points are from Ref. [11], obtained from numerical solutions of either a microscopic tight binding Anderson Hamiltonian with random site energies, or from a network of disordered quantum wires, for various values of disorder. The disorder is weakest for the squares, and strongest for the diamonds. The solid lines are from the potential (5) in the text, for $\beta=2.30$, 9.04, and 27.07 chosen to fit the squares, triangles, and diamonds, respectively. The general agreement shows that the model defined by (5) describes a qualitatively appropriate phenomenological model for disordered conductors, with β increasing with increasing disorder.

potential V was taken to be a simple quadratic v^2 in the variable v, in which the density of eigenvalues is uniform; in this variable the two-particle repulsion term turns out to be of the form $|\cosh v_i - \cosh v_j|^{\alpha}$. However, in the variable $x = \frac{1}{2}(\cosh v - 1)$ where the interaction term becomes of the standard form $|x_i - x_j|^{\alpha}$, the potential has the characteristic $(\ln x)^2$ behavior for large x. It is therefore important to explore whether the spectral statistics of a model with this particular large x behavior of the potential (the statistics turns out to be insensitive to the particular power law at small x) shows any deviation from the Wigner statistics, as expected for disordered systems. In the present work we consider a model which we show to be qualitatively similar, containing a single parameter, and exactly solvable. We then show that indeed such a potential gives rise to transition from the universal Wigner distribution to a Poisson-like spacing distribution, very similar to those seen numerically for microscopic tight binding models [8] across the metal-insulator transition.

Let us consider the potential [12]

$$V(\chi) = \sum_{n=0}^{\infty} \ln[1 + 2q^{n+1}\cosh(2\chi) + q^{2n+2}], \quad x = \sinh\chi, \tag{5}$$

where the eigenvalues x are real, between $-\infty$ and $+\infty$, and the parameter $0 < q < 1$. The general behavior of this potential for $x > 0$ is shown in Fig. 1 by the solid curves, for various values of $\beta \equiv \ln(1/q)$ (β turns out to be a more physical parameter, increasing with increasing disorder). The general agreement with the shapes obtained numerically from the microscopic models (shown by discrete points in the same figure) shows that the potential of Eq. (5) indeed defines at least a qualitatively correct [13] phenomenological model for disordered conductors [3,10,11], with the parameter β related to disorder. The advantage of this apparently more complicated model over the $a[\ln(1+bx)]^2$ potential is that it contains a single parameter, and is exactly solvable because the associated orthogonal polynomials defined by Eq. (2) are explicitly known. They are the so-called q Hermite polynomials [14] [in the variable $\sinh(x)$]. The asymptotic form ($N \to \infty$) for the kernel in this case is given by [14]

$$K(x,y) = e^{-[V(x)+V(y)]/2}\frac{Q(x,y;q)}{[(q;q)]^2 \ln(1/q)},$$
$$Q(x,y;q) = (-qe^{x+y};q)(-qe^{-x-y};q) \times (qe^{x-y};q)(qe^{-x+y};q), \tag{6}$$

where we have used the notation

$$(a;q) = \prod_{n=0}^{\infty}(1 - aq^n). \tag{7}$$

(Note that our definition of the kernel differs from Ref. [14] by the exponential prefactor as well as normalization, and the above polynomials are called q^{-1} Hermite in Ref. [14].) In order to facilitate a comparison with the Wigner distribution, for which the average spacing between adjacent eigenvalues is unity, it is necessary to rescale the eigenvalues by their average spacing. This can in general be shown to be $1/K(x,x)$ and in our case turns out to be constant equal to β. Upon rescaling of the eigenvalues, it is necessary to perform a corresponding normalization of the kernel. Then, using the scaled variables $\zeta = x/\beta$, $\eta = y/\beta$, and infinite product representations of theta functions [15] ϑ_i, the kernel in Eq. (6) can be rewritten in the form

$$K(\zeta,\eta) \propto \Omega(\beta\zeta,\beta\eta)\Theta_4(\zeta,\eta;p)\frac{\vartheta_1(\pi(\zeta-\eta);p)}{\sinh[(\zeta-\eta)\beta/2]},$$
$$\Omega(\zeta,\eta) = \frac{\sqrt{\cosh\zeta}\sqrt{\cosh\eta}}{\cosh[(\zeta+\eta)/2]}, \tag{8}$$
$$\Theta_4(\zeta,\eta;p) = \frac{\vartheta_r(\pi(\zeta+\eta);p)}{\sqrt{\vartheta_4(2\pi\zeta;p)}\sqrt{\vartheta_4(2\pi\eta;p)}},$$

where the parameter p is defined as $p = \exp(-2\pi^2/\beta)$. To look at the spacing distribution, we now restrict ourselves to the region $\zeta \approx \eta$. Although the kernel in Eq. (8) is not translationally invariant, the appropriately normalized kernel in the bulk of the spectrum, except very close to the origin, is approximately given by

$$\bar{K}(\zeta,\eta;\beta) \approx \frac{\beta}{2\pi}\frac{\sin[\pi(\zeta-\eta)]}{\sinh[(\zeta-\eta)\beta/2]}, \tag{9}$$

which is translationally invariant [16]. Let us compare Eq. (9) with the conventional random matrix result [4,17]

$$\bar{K}_0(\zeta,\eta) = \sin[\pi(\zeta-\eta)]/\pi(\zeta-\eta) \tag{10}$$

valid in the bulk of the spectrum. Here ζ and η are the corresponding scaled variables, so that the average spacing is unity. Note that $\bar{K}_0$ has no explicit dependence on any parameter once the eigenvalues have been expressed in units of average spacing. This is the source of universality in the corresponding spacing distribution $p(t)$, which is given by the well-known "Wigner" or more specifically for our unitary model the "GUE" (Gaussian unitary ensemble) distribution.

The spacing distribution $p(t)$ for the GUE increases quadratically for small spacing t, has a peak at $t \approx 1$, and decreases exponentially for large t with an exponent which is also quadratic in t (see GUE curve in Fig. 2). Our kernel $\bar{K}(\zeta,\eta;\beta)$ reduces to $\bar{K}_0(\zeta,\eta)$ in the limit $\beta \to 0$ and therefore recovers all the GUE results in this limit. For finite β, and for spacings $|\zeta - \eta| \le t \ll 1/\beta$ (note that the kernel is restricted to an interval t), the hyperbolic function can still be expanded, β scales away and we get back Eq. (10), and the universal GUE behavior for $t \to 0$, namely, $p(t) \propto t^2$ follows. However, there is now a rather large region where the spacing can be of the order unity but at the same time much larger than $1/\beta$ for large enough β. In this regime the parameter β cannot be scaled away, and the spacing distribution obtained from it becomes sensitive to the cutoff provided by the hyperbolic function. Thus we expect that the peak is shifted from $t \approx 1$ to $t \propto 1/\beta$. In Fig. 2 we show the spac-

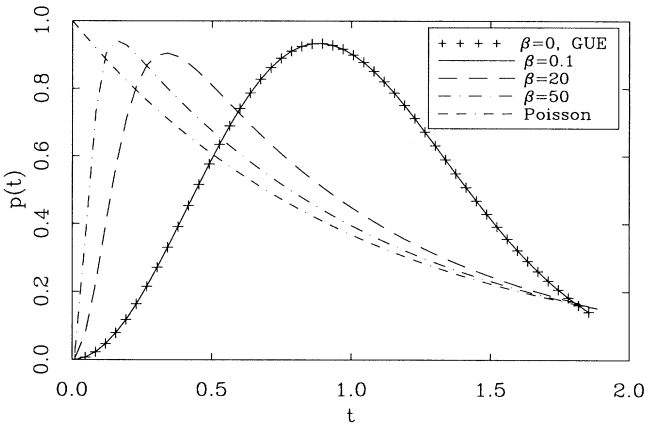

FIG. 2. The spacing distribution $p(t)$ as a function of the spacing t between nearest eigenvalues for various values of the parameter β. For comparison, the universal GUE distribution, which coincides with the $\beta=0$ curve, and the Poisson (exponential) distribution are also included.

ing distribution for nearest neighbor eigenvalues obtained from a numerical evaluation of the determinant in Eq. (4) using the approximate kernel of Eq. (9), for various values of β, which interpolates between a Wigner distribution ($\beta \to 0$) and a distribution which becomes increasingly more Poisson-like (see Fig. 2) with increasing β, i.e., increasing disorder. This is precisely the transition seen in numerical simulations of finite disordered systems going through a metal-insulator transition [8]. (In fact comparing with results of Ref. [10], we can identify β with the ratio of system size and localization length in the insulating side.) A similar transition is also seen in simulations describing onset of chaos [9], and it would be interesting to see if such a natural connection exists there as well.

In this calculation we restricted ourselves for simplicity to the case of unitary ensembles only. We expect similar deviations for the orthogonal and symplectic cases also. More importantly we chose a particular potential related to the q Hermite as opposed to, e.g., q Laguerre polynomials [18] considered in Ref. [10], which has a different range of eigenvalues. However, the transition from a Wigner to a Poisson-like distribution is a consequence of the large x behavior of the potential. Since the physical motivation for the introduction of the new family came mainly from the region of large x, and q Laguerre and q Hermite polynomials (and the associated potentials) share the same asymptotic behavior, we expect that the family of q polynomials in general [19] would describe matrix ensembles with similar spectral properties.

In summary, we found that the appropriate random matrix model for disordered conductors belongs at least qualitatively to a new family of random matrices. The nearest neighbor spacing distribution for the eigenvalues of these matrices does not follow the universal Wigner form associated with the conventional random matrix models. Instead, the spacing distribution shows a transition from the highly correlated Wigner to an uncorrelated Poisson form with increasing disorder, as seen in numerical solutions of microscopic models. We note that similar transitions also occur in dynamical systems near the regular-chaotic transition.

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[13] Note that we have considered a model where the eigenvalues range from $-\infty$ to $+\infty$; the transfer matrix models discussed for disordered systems have eigenvalues restricted to $+ve$ values only. However, because the transition from a Wigner to a Poisson-like distribution is a consequence of the large x behavior of the potential, we expect that this difference will be irrelevant for the qualitative behavior of the spectra in the bulk; in fact, the classical Laguerre and Hermite measures with such a difference are known to lead to the same spacing distribution in the bulk [17].

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