Order $\alpha^4 R_\infty$ Corrections to the Fine-Structure Splitting of Positronium P levels

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The order $\alpha^4 R_\infty$ corrections to the fine-structure splitting of positronium P levels are found. The calculation is reduced to the ordinary perturbation theory for the nonrelativistic Schrödinger equation. The perturbation operators have the Breit-type structure and are obtained by calculating on-mass-shell diagrams. The correction found is -0.64 MHz for the 2^3P_0-2^1P_1 energy splitting.

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Precision measurements of positronium structure provide a unique test of quantum electrodynamics. The typical accuracy reached in the measurements of the positronium 2^3S_1-2^3P_J $(J = 0,1,2)$, 1^3S_1-2^3S_1, and 2^3S_1-2^1P_1 intervals constitutes few MHz [1–6]. The account for the order $\alpha^3\log(1/\alpha)R_\infty$ and $\alpha^3 R_\infty$ corrections [7–9] is now insufficient for the comparison of quantum electrodynamics with those experimental data. Here $R_\infty = 109\,737.315\,682\,7(48)$ cm^{-1} is the Rydberg constant.

The order $\alpha^4\log(1/\alpha)R_\infty$ corrections to the positronium levels were calculated recently by Fell [10] and then by us [11]. This shift exists in the S states only and scales as n^{-3}. It is in fact a relativistic correction [11], but its logarithmic structure allows one to treat the relativistic effects as a perturbation when deriving the result. However, if one tries to go beyond the logarithmic approximations, the logarithmic integrals which are cut off at the electron mass m should be treated exactly, and the problem becomes extremely tedious.

Fortunately, for states of higher angular momenta, $L > 0$, the situation is better since their nonrelativistic wave functions fall off more rapidly at small distances, or at large momenta. Therefore, the integrals arising when treating the relativistic effects as perturbation, converge in the nonrelativistic region which makes the problem quite tractable. The main complication (underestimated by us at the beginning of the work) is of the "bookkeeping" nature. Analogous results for the electron-electron interaction in helium were obtained numerically in Ref. [12].

Here we present the results of our analytical calculations for the fine-structure splitting of positronium P levels. In the case $n = 2$ they can be directly compared with the data extracted from the experimental results of Refs. [2,5,6,13]. The corresponding investigation of the overall shift of nP levels is somewhat more complicated technically, and is discussed in our more detailed paper [14].

Let us start with the correction generated by the v^4/c^4 term in the Breit Hamiltonian due to the Coulomb exchange. This energy correction equals

$$\delta E_C = \epsilon_n\left\{-\left[\frac{3}{320}\left(1-\frac{13}{12n^2}\right)+\frac{1}{384}\left(1-\frac{1}{n^2}\right)\right]\mathbf{L}\cdot\mathbf{S} +\frac{1}{9600}\left(1-\frac{3}{2n^2}\right)[2(\mathbf{L}\cdot\mathbf{S})^2+\mathbf{L}\cdot\mathbf{S}]+\frac{1}{4800}\left(1-\frac{3}{2n^2}\right)S(S+1)\right\}, \quad (1)$$

where $\epsilon_n \equiv m\alpha^6/n^3$.

Here and below we omit systematically all the terms which depend neither on the total spin S, nor on the total angular momentum J.

Let us consider now the contribution to the energy due to the exchange by one magnetic photon. The necessary contribution is produced here by the v^2/c^2 corrections to the currents which gives

$$\delta E_{\text{curr}} = \epsilon_n\left\{-\left[\frac{1}{40}\left(1-\frac{13}{12n^2}\right)+\frac{1}{320}\left(1-\frac{2}{3n^2}\right)\right]\mathbf{L}\cdot\mathbf{S} +\left[\frac{3}{400}\left(1-\frac{13}{12n^2}\right)-\frac{1}{800}\left(1-\frac{2}{3n^2}\right)\right][2(\mathbf{L}\cdot\mathbf{S})^2+\mathbf{L}\cdot\mathbf{S}]-\frac{1}{80}\left(1-\frac{1}{n^2}\right)S(S+1)\right\}. \quad (2)$$

One more energy correction originates from the retardation effects in the magnetic interaction:

$$\delta E_{\text{ret}} = \epsilon_n\left\{\frac{1}{480}\left(13-\frac{2}{n^2}\right)\mathbf{L}\cdot\mathbf{S}-\frac{1}{4800}\left(13-\frac{2}{n^2}\right)[2(\mathbf{L}\cdot\mathbf{S})^2+\mathbf{L}\cdot\mathbf{S}]+\frac{1}{600}\left(13-\frac{2}{n^2}\right)S(S+1)\right\}. \quad (3)$$

Let us pass over now to the corrections generated by the Feynman diagrams with both fermions staying always in positive-energy states and magnetic quantum (wavy line) crossed by one and two Coulomb (dashed) lines [see Figs.

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1(a) and (b)]. Those corrections are, respectively,

$$\delta E_{MC} = \epsilon_n \left\{ -\frac{1}{24}\mathbf{L}\cdot\mathbf{S} + \left[\frac{1}{240} + \frac{1}{800}\left(1-\frac{2}{3n^2}\right)\right]\left[2(\mathbf{L}\cdot\mathbf{S})^2 + \mathbf{L}\cdot\mathbf{S}\right] - \left[\frac{1}{30}+\frac{1}{100}\left(1-\frac{2}{3n^2}\right)\right] S(S+1)\right\}, \quad (4)$$

$$\delta E_{MCC} = \epsilon_n \left\{ \frac{1}{48}\mathbf{L}\cdot\mathbf{S} - \frac{1}{480}\left[2(\mathbf{L}\cdot\mathbf{S})^2 + \mathbf{L}\cdot\mathbf{S}\right] + \frac{1}{60}S(S+1)\right\}. \quad (5)$$

Two more contributions are due to Z-type diagrams (where one fermion is in a negative-energy intermediate state) with one and two magnetic quanta [see Figs. 2(a) and (b)]:

$$\delta E_{M-} = \epsilon_n \left\{ -\frac{1}{80}\left(1-\frac{2}{3n^2}\right)\mathbf{L}\cdot\mathbf{S} + \frac{1}{200}\left(1-\frac{2}{3n^2}\right)\left[2(\mathbf{L}\cdot\mathbf{S})^2 + \mathbf{L}\cdot\mathbf{S}\right] + \frac{1}{100}\left(1-\frac{2}{3n^2}\right) S(S+1)\right\}, \quad (6)$$

$$\delta E_{MM} = -\epsilon_n \frac{1}{80}\left(1-\frac{2}{3n^2}\right)\mathbf{L}\cdot\mathbf{S}. \quad (7)$$

The sum of all those corrections can be written in a compact form:

$$\delta E^{(1)} = \frac{\epsilon_n}{1920}\left[-\left(90-\frac{83}{n^2}\right)\mathbf{L}\cdot\mathbf{S} + \left(46-\frac{43}{n^2}\right)(\mathbf{L}\cdot\mathbf{S})^2 - \left(14-\frac{17}{n^2}\right)S(S+1)\right]. \quad (8)$$

The next class of the order $\alpha^4 R_\infty$ corrections originates from the iteration of the usual Breit Hamiltonian V of the second order in v/c. In the case of P states of interest to us it can be conveniently presented as

$$V = V_0 + V_1, \quad (9)$$

where

$$V_0 = \frac{\alpha^2}{4mr^2} - \frac{3}{4m}\left(\frac{p^2}{m}\frac{\alpha}{r} + \frac{\alpha}{r}\frac{p^2}{m}\right) + \frac{\alpha}{m^2r^3} \quad (10)$$

is spin independent, and

$$V_1 = \frac{3}{2}\frac{\alpha}{m^2r^3}\left[\mathbf{L}\cdot\mathbf{S} + S_mS_n\left(n_mn_n - \tfrac{1}{3}\delta_{mn}\right)\right], \quad \mathbf{n} = \frac{\mathbf{r}}{r}. \quad (11)$$

The splitting among P levels is contributed to by the cross term between V_0 and V_1, as well as by the correction of second order in V_1. Those contributions are, respectively,

$$\delta E_{01}^{(2)} = \epsilon_n \frac{\kappa}{720}\left(\frac{211}{4} + \frac{45}{n} - \frac{29\times 27}{8n^2}\right),$$
$$\delta E_{11}^{(2)} = -\epsilon_n \frac{\kappa^2}{512\times 15}\left(227 + \frac{90}{n} - \frac{108}{n^2}\right),$$
$$\kappa = \tfrac{1}{5}\left[-2(\mathbf{L}\cdot\mathbf{S})^2 + 4\mathbf{L}\cdot\mathbf{S} + \tfrac{4}{3}S(S+1)\right]. \quad (12)$$

In fact, $\delta E_{11}^{(2)}$ refers to the second-order correction due to the ^3P_J intermediate states. However, the tensor part,

$$\frac{3}{2}\frac{\alpha}{m^2r^3}S_mS_n\left(n_mn_n - \tfrac{1}{3}\delta_{mn}\right),$$

of V_1 admixes the states $^3F_2$ to $^3P_2$. The corresponding energy correction is

$$\delta E_F^{(2)} = -\frac{\epsilon_n}{800}\left(1-\frac{7}{10n^2}\right). \quad (13)$$

Curiously enough, even the true radiative corrections of the $\alpha^4 R_\infty$ order to the fine splitting discussed can be presented in a simple form practically without special calculation. They can be easily demonstrated to be confined here to anomalous magnetic moment contribution to the usual Breit Hamiltonian. If the electron magnetic moment is presented as

$$\mu = \frac{e}{2m}\left[1 + g_1\frac{\alpha}{\pi} + g_2\left(\frac{\alpha}{\pi}\right)^2\right],$$
$$g_1 = 0.5, \quad g_2 = -0.328, \quad (14)$$

then the true radiative correction is

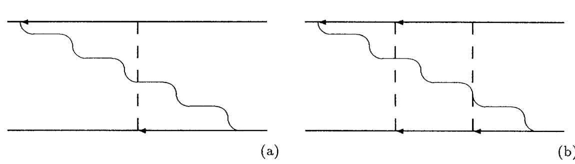

FIG. 1. Magnetic interaction with the Coulomb exchange.

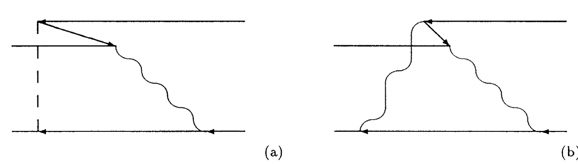

FIG. 2. Z-type magnetic exchange.

TABLE I. Fine-structure intervals between $2^{2S+1}P_J$ levels (in MHz).

	Experiments		Theory	
	Michigan [2,5]	Mainz [6,13]	Total	$\alpha^4 R_\infty$ contribution
$E\left(^3P_2\right)$-$E\left(^3P_0\right)$	9884.5±10.5	9875.27 ±4.44	9871.54	0.66
$E\left(^3P_1\right)$-$E\left(^3P_0\right)$	5502.8±10.9	5487.23 ±4.50	5485.84	0.60
$E\left(^1P_1\right)$-$E\left(^3P_0\right)$	7323.1±16.5	7319.65 ±7.64	7312.88	0.64

$$\delta E_{\rm rad} = \frac{\epsilon_n}{\pi^2}\left[\frac{g_2}{24}\mathbf{L}\cdot\mathbf{S} - \frac{g_1^2+2g_2}{80}\left[2(\mathbf{L}\cdot\mathbf{S})^2+\mathbf{L}\cdot\mathbf{S}\right] + \frac{g_1^2+2g_2}{60}S(S+1)\right]. \tag{15}$$

Let us summarize the numerical values of the $\alpha^4 R_\infty$ corrections to the energies of 2^3P_J levels as counted off the 2^1P_1 one. They are 0.02 MHz for $J=2$; -0.04 MHz for $J=1$; and -0.64 MHz for $J=0$. The shift of the singlet state 2^1P_1 itself is equal numerically to 0.06 MHz (see Ref. [14]).

Including $\alpha^2 R_\infty$ and $\alpha^3 R_\infty$ terms we obtain the fine-structure intervals between $2^{2S+1}P_J$ levels. In Table I these theoretical values are compared with the transition frequencies extracted from the results of two recent experiments [2,5,6,13] (all systematic and statistical errors are added quadratically when extracting the experimental numbers).

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