Scaling Behavior of Chaotic Systems with Riddled Basins

Edward Ott,[1,*] John C. Sommerer,[2] James C. Alexander,[1,†] Ittai Kan,[3] and James A. Yorke[1,†,‡]
[1] University of Maryland, College Park, Maryland 20742
[2] M. S. Eisenhower Research Center, Johns Hopkins University Applied Physics Laboratory, Laurel, Maryland 20723
[3] Department of Mathematics, George Mason University, Fairfax, Virginia 22030
(Received 30 July 1993)

Recently it has been shown that there are chaotic attractors whose basins are such that *every* point in the attractor's basin has pieces of another attractor's basin arbitrarily nearby (the basin is "riddled" with holes). Here we report quantitative theoretical results for such basins and compare with numerical experiments on a simple physical model.

PACS numbers: 05.45.+b

Recently it has been shown [1,2] that chaotic systems with a simple, very common type of symmetry can display a striking new kind of behavior. In particular, these systems may have an attractor whose basin of attraction is such that *every* point in the basin has pieces of another attractor's basin arbitrarily nearby [3]. That is, if $\mathbf{r}_0$ is *any* point in the first attractor's basin, then the phase space ball of radius ϵ centered at $\mathbf{r}_0$ has a *nonzero* fraction of its volume lying in a another attractor's basin, and this is so no matter how small ϵ is. Thus there is *always* a positive probability that an arbitrarily small uncertainty in $\mathbf{r}_0$ will put the initial condition in another attractor's basin [4]. We say that the first basin is *riddled* by the second basin.

This paper reports quantitative theoretical results [5] for riddled basins and compares these results with numerical experiments on a simple mechanical system [2]. The theoretical results apply near the critical point where a riddled basin first appears as a system parameter is varied. In this parameter regime the behavior becomes universal [6] in the sense that it is controlled by a few gross system variables. We emphasize, however, that the *qualitative* behavior found persists away from the transition.

A general set of conditions under which riddled basins can occur are as follows: (i) There is an *invariant* subspace M whose dimension d_M is less than that of the phase space $d_{\rm ps}$. (ii) The dynamics on the invariant subspace M has a chaotic attractor A for initial conditions on M. (iii) For typical orbits on A the Lyapunov exponents for infinitesimal perturbations in the directions transverse to M are negative, so that A is also an attractor in the full $d_{\rm ps}$ dimensional phase space. (iv) At least one of the transverse Lyapunov exponents, although negative for almost any orbit on A, experiences finite time fluctuations that are positive. (v) There is another attractor not in M.

In order to illustrate the above, we refer to an example. In particular, we consider the motion of a point particle of unit mass moving in a two-dimensional potential $V(x,y)$ subject to dynamic friction (coefficient ν) and sinusoidal forcing,

$$d\mathbf{r}/dt^2 = -\nu d\mathbf{r}/dt - \nabla V + f_0 \sin(\omega t)\mathbf{x}_0, \qquad (1)$$

where $V(x,y) = (1-x^2)^2 + (x+\bar{x})y^2$, $\mathbf{x}_0$ is a unit vector in the x direction and ν, f_0, ω, and $\bar{x}$ are parameters. The phase space is five dimensional ($d_{\rm ps} = 5$) with coordinates x, dx/dt, y, dy/dt, and $\theta = (\omega t) \bmod 2\pi$. From the symmetry of the potential [i.e., $V(x,y) = V(x,-y)$], we see that the dynamics is invariant with respect to $(y, dy/dt) \to (-y, -dy/dt)$, and, as a consequence, $y = dy/dt = 0$ specifies an invariant three-dimensional hyperplane [the subspace M in condition (i)] in the full five-dimensional phase space [7]. In this invariant plane, points are specified by these coordinates, x, dx/dt, and θ, and the dynamics is described by the forced double-well Duffing equation,

$$d^2x/dt^2 + \nu dx/dt - 4x(1-x^2) = f_0 \sin \omega t, \qquad (2)$$

which, for appropriate values of the parameters (ν, f_0, ω) has a chaotic attractor A in the invariant subspace M [condition (ii)]. The set A will also be an attractor for the full five-dimensional dynamical system, Eq. (1), if condition (iii) holds. Figure 1 shows a plot of the larger of the two transverse Lyapunov exponents, denoted $h_\perp$, as a function of the parameter $\bar{x}$ with the other parameters held fixed at $\nu = 0.05$, $f_0 = 2.3$, and $\omega = 3.5$ (these parameter values are used throughout the rest of this paper). We see from Fig. 1 that $h_\perp < 0$ [condition (iii)] for $\bar{x} > \bar{x}_c = 1.7887\ldots$. This behavior can be understood as resulting from the term $(x+\bar{x})y^2$ in the potential. When $x+\bar{x} > 0$, there is a y^2 potential well focusing orbits onto the invariant plane; when $x+\bar{x} < 0$, the well becomes an antiwell. Thus focusing is expected to prevail for large enough $\bar{x}$ (i.e., for $\bar{x} > \bar{x}_c$).

While the long time limit involved in the calculation of $h_\perp$ is negative for $\bar{x} > \bar{x}_c$, there will also be epochs of time during which a typical chaotic orbit spends an abnormally long time in the defocusing region. Thus, if such an orbit is given a small perturbation in the direction transverse to M, it temporarily moves in the direction away [8,9] from M [condition (iv)]. A quantity (denoted D below) characterizing the fluctuations in finite time estimates of $h_\perp$ can be defined as follows. Consider a very large number N of randomly chosen initial conditions on A. For each initial condition $\mathbf{r}_i$ ($i = 1, 2, \ldots, N$) calculate

0031-9007/93/71(25)/4134(4)$06.00

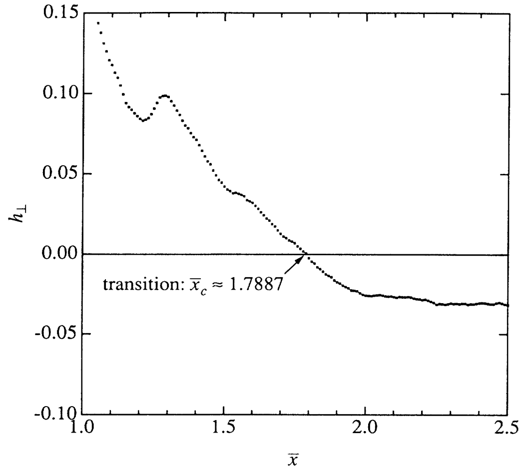

FIG. 1. $h_\perp$ versus $\bar{x}$.

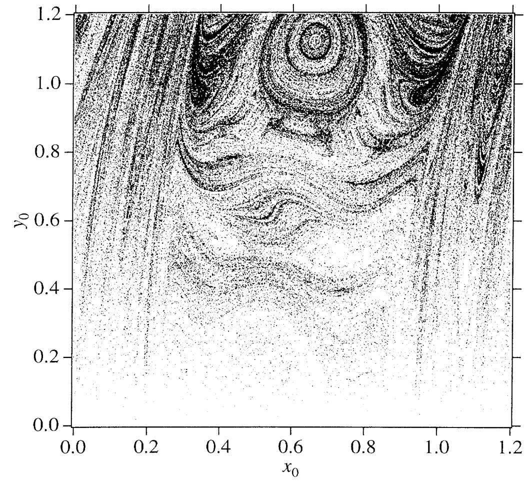

FIG. 2. Initial conditions in the $|y| = \infty$ basin are plotted as black dots.

the finite time transverse exponent $\tilde{h}_\perp(\mathbf{r}_i, t)$ over the time interval zero to t. In the limit $t \to \infty$ all the $\tilde{h}_\perp(\mathbf{r}_i, t)$ approach $h_\perp$ (with probability 1). For finite time, however, there is a spread $\langle(\tilde{h}_\perp - \langle\tilde{h}_\perp\rangle)^2\rangle^{1/2}$, where $\langle\cdots\rangle$ denotes an average over i. This spread approaches zero inversely with $\sqrt{n}$. We now define the quantity D by the relation ($n \gg 1$)

$$\langle(\tilde{h}_\perp - \langle\tilde{h}_\perp\rangle)^2\rangle \cong 2D/n. \tag{3}$$

For example, a plot of $\langle(\tilde{h}_\perp - \langle\tilde{h}_\perp\rangle)^2\rangle n^2/2$ versus n obtained numerically for Eq. (1) with $\bar{x} > \bar{x}_c$ yields data that are well fitted by a straight line, whose slope we take to be an estimate of D.

In addition to the $y = dy/dt = 0$ attractor, for appropriate initial conditions, Eq. (1) also has orbits which diverge to $|y| = \infty$. We regard $|y| = \infty$ as a second attractor [condition (v)]. Figure 2 shows a two-dimensional slice ($dx/dt = dy/dt = \theta = 0$) through the five-dimensional riddled basin of the $y = 0$ attractor for $\bar{x} = 1.9$. Each initial condition on a 1024×1024 grid with $x(0)$ and $y(0)$ given by the horizontal and vertical axes was followed numerically. A black dot was plotted on grid points leading to $|y| = \infty$, while no dot was plotted for orbits tending to the attractor on $y = dy/dt = 0$. Blow-ups on a sufficiently finer scale of any small region containing a point going to the $y = 0$ attractor always show black dots, thus supporting our contention that the $y = 0$ basin is riddled. (Note, however, that the $|y| = \infty$ basin is not riddled: for almost any point in the black region, a blowup about the point at large enough magnification will yield a solid black picture.)

A general theory for cases like that in our example will be reported elsewhere [6]. In what follows we quote two predictions from Ref. [6] and then compare them with numerical experiments.

Measure of the basins.—Say we draw a horizontal line at $y(0) = y_0 =$ const in Fig. 2. Now say we evaluate the fraction P_* of the length of that line that is in the basin of the $|y| = \infty$ attractor. That fraction is predicted [6] to scale as $P_* \sim |y_0|^\eta$, where the exponent η is given by

$$\eta = |h_\perp|/D. \tag{4}$$

Thus, as the distance to the invariant manifold M becomes smaller, the fraction of the initial conditions in the basin of A approaches 1. Nevertheless, at any small nonzero distance ($|y_0| > 0$), there is always some positive fraction in the $|y| = \infty$ basin. Thus there are pieces of the $|y| = \infty$ basin that are arbitrarily close to the attractor [3] in $y = dy/dt = 0$. Hence, if noise is added to motion on the noiseless attractor, there is the possibility that the orbit is perturbed to one that goes to $|y| = \infty$. Thus arbitrarily small noise destroys the noiseless attractor. However, if the noise is small, it typically takes a very long time to see this [6]. Figure 3(a) shows results of numerical experiments (open circles) at $\bar{x} = 1.85$ for P_* versus y_0. Here P_* is numerically estimated by taking many randomly chosen initial conditions on the line $y(0) = y_0$, evolving them forward in time to determine which basin they lie in, and then taking the estimate of P_* as the fraction of these points that go to $|y| = \infty$. The solid line has a slope given by Eq. (4) with the numerically determined $h_\perp$ and D. The agreement is quite good.

Fat fractal uncertainty exponent.—The result (4) gives the measure of the basins but says nothing about the arbitrarily fine scaled riddling of the $y = dy/dt = 0$ attractor basin. In the language of Ref. [10] the riddled basin is a "fat fractal." The fine scaled riddling of the fat fractal basin can be characterized by the *uncertainty exponent* introduced for fat fractals in Ref. [11]. Again consider a horizontal line $y(0) = y_0$ in Fig. 2. Imagine that we choose a point at random on that line [i.e., we choose $x(0)$ randomly]. Now choose a second point at

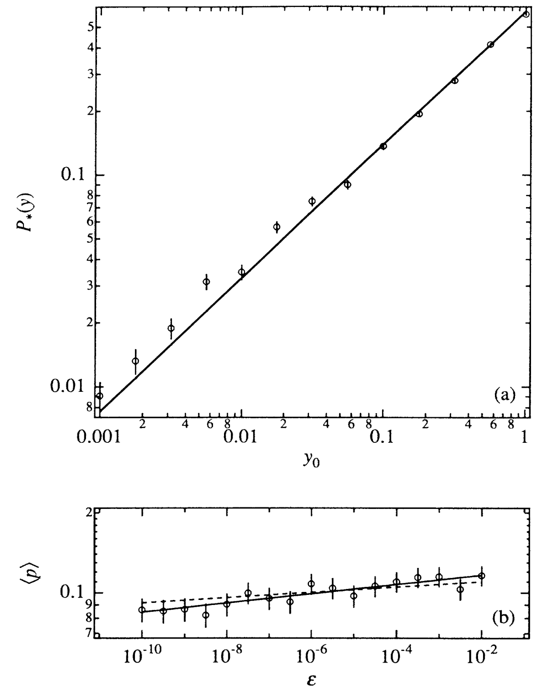

FIG. 3. (a) Data for P_* versus y_0. (b) Data for $\langle p \rangle$ versus ϵ.

random on the same line with uniform probability in the interval of length 2ϵ centered at the first point [i.e., we choose a second $x(0)$ within a distance ϵ of the first]. We ask, what is the probability that these two points are in different basins (one in the $|y| = \infty$ basin and the other in the basin of the $y = dy/dt = 0$ attractor)? We denote this probability $\langle p \rangle$ and define the uncertainty exponent ϕ by the small ϵ scaling of $\langle p \rangle$: $\langle p \rangle \sim \epsilon^{\phi}$. We can think of $\langle p \rangle$ as the probability of making an error if we attempt to predict which basin $(x(0), y(0))$ is in when $x(0)$ has a measurement uncertainty ϵ. The result of Ref. [6] for ϕ is

$$\phi = h_\perp^2 / 4 D h_\parallel , \tag{5}$$

where $h_\parallel$ is the positive Lyapunov exponent for the dynamics in M [the positive exponent computed for Eq. (2) for our example]. Figure 3(b) shows $\langle p \rangle$ versus ϵ for $\bar{x} = 1.9$ (recall $\bar{x}_c = 1.7887\ldots$) and $y(0) = 0.5$. Here $\langle p \rangle$ is numerically estimated by randomly choosing many points on the horizontal line $y(0) = 0.5$, then pairing each such point with a point randomly chosen within ϵ of it, and then integrating (1) to determine which basin each point lies in. The data in Figure 3(b) are indicated as open circles with the sampling error indicated as vertical line segments through the circles. The best fit line to the data (shown as the solid line) gives a slope of $\phi \cong 0.017$, a very small value. This value of ϕ and the scaling $\langle p \rangle \sim \epsilon^{\phi}$ correspond to a decrease of $\langle p \rangle$ by only about 27% when ϵ decreases from 10^{-2} to 10^{-10}. The uncertainty exponent predicted by Eq. (5) with the numerically determined values of $h_\perp$, $h_\parallel$, and D is about 0.009 corresponding to the dashed line in Fig. 3(b). Thus the predicted ϕ yields a decrease of $\langle p \rangle$ by about 15% for ϵ decreasing from 10^{-2} to 10^{-10}. Given the scatter evident in Fig. 3(b) and the small value of ϕ, we do not regard the discrepancy between the fitted and theoretical results as significant. The striking point, evident for both the prediction and the fit, is that the decrease in $\langle p \rangle$ is of the order of only tens of percent when ϵ decreases by 8 orders of magnitude. Thus a vast improvement in the accuracy of initial conditions does surprisingly little as far as reducing the uncertainty in determining which attractor is ultimately approached.

To illustrate the origin of Eqs. (4) and (5), we consider a simple two-dimensional map of the region $y \geq 0$, $0 \leq x \leq 1$, satisfying conditions (i)–(v). In the region $0 \leq y \leq 1$ the map is $(x_{n+1}, y_{n+1}) = (x_n/\alpha, 2y_n)$ for $0 \leq x_n < \alpha$ and $(x_{n+1}, y_{n+1}) = (x_n/(1-\alpha), y_n/2)$ for $\alpha \leq x_n \leq 1$. For $y > 1$, we imagine that the form of the map is such that orbits falling in $y > 1$ all move to an attractor in $y > 1$. The invariant line segment, $y = 0$, $0 \leq x \leq 1$, is a riddled basin chaotic attractor [6] if the perpendicular Lyapunov exponent $h_\perp = (2\alpha - 1)\ln 2$ is negative (i.e., $0 < \alpha < 1/2$). Now consider the horizontal line segment $y = y_0$, $0 \leq x \leq 1$, where $0 < y_0 < 1$. Imagine that we choose an initial condition x_0 at random with uniform probability distribution per unit length along this line segment. The probability that the resulting orbit goes to the $y > 1$ attractor is the fraction P_* of the line length in the $y > 1$ attractor's basin. Furthermore, the dynamics in x for random x_0 is such that, on any given iterate, the probability that $0 \leq x_n < \alpha$ is α, while the probability that $\alpha \leq x_n \leq 1$ is $1 - \alpha$. Referring to the y component of the map, we see that $y \to 2y$ with probability α, and $y \to y/2$ with the probability $1 - \alpha$. Let $z = \log_2(1/y)$. We now have a random walk in z in which a unit step to the right (left) has probability $1 - \alpha$ (probability α). The fraction P_* of the line segment is now given by the solution to the following standard problem: Starting at $z_0 = \log_2(1/y_0)$, what is the probability that the random walker ever reaches $z = 0$ (corresponding to $y = 1$)? For small $h_\perp$ (i.e., $1/2 - \alpha > 0$ small), this probability is proportional to y_0^{η} with η given by Eq. (4), and D the diffusion coefficient of the random walk. This model can also be used [6] to obtain Eq. (5). We argue in Ref. [6] that Eqs. (4) and (5) are universal near the transition, and this is consistent with our numerical results (Fig. 3) for the ordinary differential equation system (1).

This work was supported by the Office of Naval Research (Physics), by the Department of Energy, by the U.S. Naval Space and Warfare Systems Command, by the Air Force Office of Scientific Research, and by the National Science Foundation.

---

* Department of Electrical Engineering, Department of

Physics, Laboratory for Plasma Research, and Institute for Systems Research
† Department of Mathematics.
‡ Institute for Physical Science and Technology.

[1] J. C. Alexander, I. Kan, J. A. Yorke, and Z. You, Int. J. Bif. Chaos **2**, 795 (1992); I. Kan (to be published).
[2] J. C. Sommerer and E. Ott, Nature (London) **365**, 136 (1993).
[3] Here we use the definition of an attractor given by Milnor: An attractor has a basin of attraction of positive Lebesgue measure (volume) in phase space and is a compact set with a dense orbit [J. Milnor, Commun. Math. Phys. **99**, 177 (1985)].
[4] We emphasize the difference between this new kind of basin and previously investigated basins possessing interior phase space volume and a fractal boundary [e.g., S. W. McDonald *et al.*, Physica (Amsterdam) **17D**, 125 (1985)], for which small uncertainty can only lead to a transfer of basins when $\mathbf{r}_0$ is near the boundary.
[5] Further details will be provided in a future publication. [E. Ott, J. C. Alexander, I. Kan, J. C. Sommerer, and J. A. Yorke (to be published)].
[6] Ott *et al.* (Ref. [5]).
[7] In other problems, other kinds of symmetry can lead to the existence of an invariant manifold, and such cases are also very common.
[8] Such epochs become longer and more common as $\bar{x}$ is brought nearer to $\bar{x}_c$.
[9] This also implies that there is a collection of nontypical (i.e., zero natural measure) orbits on A for which the transverse Lyapunov exponent is positive.
[10] J. D. Farmer, Phys. Rev. Lett. **55**, 351 (1985).
[11] C. Grebogi, S. W. McDonald, E. Ott, and J. A. Yorke, Phys. Lett. A **110**, 1 (1985).