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# Generalized Symmetries and $w_\infty$ Algebras in Three-Dimensional Toda Field Theory

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After establishing a formal theory for getting solutions of one type of high-dimensional partial differential equation, two sets of generalized symmetries of the 3D Toda theory, which arises from a particular reduction of the 4D self-dual gravity equation, are obtained concretely by a simple formula. Each set of symmetries constitutes a generalized $w_\infty$ algebra which contains three types of the usual $w_\infty$ algebras as special cases. Some open questions are discussed.

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$W_\infty$ is a particular generalization of the Virasoro algebra which contains conformal spins $2,3,\ldots,\infty$ [1,2]. The Virasoro algebra is a special subalgebra of $W_\infty$ containing only the spin-2 field: the two-dimensional (2D) energy-momentum tensor. The importance of the Virasoro algebra in string theory, integrable models [say, Korteweg–de Vries (KdV) hierarchy], 2D gravity, and conformal field theory is well known [3,4]. It is natural to search for a higher spin generalization of the Virasoro algebra which leads to the $W_\infty$ algebra. Recently one has also known that $W_\infty$ and a certain contraction of $W_\infty$, called $w_\infty$ [5], play an important role in some different areas of physics, such as the $\mathrm{sl}(\infty)$ Toda theory [6,7], integrable models (like the Kadomtsev-Petviashvili (KP) equation [8,9]), membrane theory [10], and $W$ string and $W$ gravity theories [11]. $w_\infty$ algebra is defined by the Lie bracket

$$[w_{m_1}^{n_1},w_{m_2}^{n_2}]=[(m_2-1)n_1-(m_1-1)n_2]w_{m_1+m_2-2}^{n_1+n_2} \quad (m\geq 2)\,. \tag{1}$$

The "classical" $w_\infty$ and "quantum" $w_\infty$ were realized using two-dimensional bosonic [12] and fermionic [13] fields. There exist various realization methods of $w_\infty$. For instance, $w_\infty$ emerges as a symmetry group in the study of self-dual gravity, which can be formulated as the $\mathrm{sl}(\infty)$ Toda theory [6]. And $w_\infty$ also appears as a symmetry group of the KP hierarchy.

In Ref. [14], David *et al.* proved that the KP equation has additional symmetries which obey a Kac-Moody-Virasoro based on a subalgebra of $\mathrm{Vir}\oplus\hat{\mathrm{sl}}(5,R)$. More recently, the author found that the KP equation possesses more fruitful symmetries [9] by using the extended master-symmetry approach [15]. The Kac-Moody-Virasoro [14] and $w_\infty$ are special subalgebras of the generalized symmetry algebra of the KP equation [9]. Now a natural question is the following: Is there such a type of generalized symmetry algebra in other physical fields? Obviously, the first work we hope to do is to answer whether the more generalized symmetry algebras (which contain $w_\infty$ as a special subalgebra) can be found in the $\mathrm{sl}(\infty)$ Toda theory.

In the $N\to\infty$ limit, the 2D $\mathrm{sl}(N+1)$ Toda field theory, described by the equations [16]

$$\partial\bar\partial\phi_i=\frac{1}{4}\sum_{\{\alpha\}}\alpha_i\exp(\alpha\cdot\phi),\quad \partial=\frac{\partial}{\partial Z}\,,\qquad \bar\partial=\frac{\partial}{\partial\bar Z},\quad Z,\quad \bar Z=x\pm t\,, \tag{2}$$

where $\alpha$ are the simple roots of $\mathrm{sl}(N+1)$, becomes the 3D (2+1 dimensional) Toda equation [7,17]

$$\partial\bar\partial\phi+\exp(-\partial_s^2\cdot\phi)=0\,, \tag{3}$$

with

$$u(Z,\bar Z,s_0+i\Delta)\equiv\Delta\phi_i(Z,\bar Z),\quad \Delta=\frac{s_{i+1}-s_i}{N}, \tag{4}$$

and $\phi(Z,\bar Z,s)=-\partial_s^{-1}u(Z,\bar Z,s)$. The $w_\infty$ currents of



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(3) have been studied in Ref. [7], where $w_3$ and $w_4$ currents are given explicitly. In this Letter we study the symmetry algebra in an alternative way.

Next we develop a simple direct formal method for searching generalized symmetries of a type of high-dimensional partial differential equations. The generalized symmetries and Lie algebras for the 3D Toda equation (2) are then given.

*A formal theory for searching symmetries in high dimensions.*— We consider a generalized $(M+1)$-dimensional partial differential equation (PDE) with the form

$$u_{tx_1}=K(t,x_1,x_2,\ldots,x_M,u,u_{x_1},u_{x_2},\ldots)\equiv K(u),\quad M\geq 2, \tag{5}$$

where $K(u)$ is an arbitrary function of the space time and space derivatives but is not dependent on time derivatives. The KP equation and the 3D Toda equation (with "time" $Z$ or $\bar{Z}$) are two special examples. A symmetry of (5) is defined as a solution of the linearized equation of (5) [18],

$$\sigma_{tx_1}=K'\sigma\equiv\frac{\partial}{\partial\epsilon}K(u+\epsilon\sigma)|_{\epsilon=0}, \tag{6}$$

which means (5) is form invariant under the transformation

$$u\to u+\epsilon\sigma \tag{7}$$

with infinitesimal parameter $\epsilon$. Now we search for the solutions of (6) which have the form

$$\sigma(f)=\sum_{k=0}^{\infty}f^{(-k)}\sigma[k], \tag{8}$$

where $f=f(t)$ is an arbitrary function of $t$, $f^{(-k)}=\partial_t^{(-k)}f$, and $\partial_t^{-1}=\int^t dt$. Substituting (8) into (6) directly we have

$$\sum_{k=0}^{\infty}f^{(-k+1)}\sigma_{x_1}[k]+\sum_{k=1}^{\infty}f^{(-k+1)}\sigma_{tx_1}[k-1]=\sum_{k=1}^{\infty}f^{(-k+1)}K'\sigma[k-1]. \tag{9}$$

Since $f$ is an arbitrary function of $t$, deleting the coefficients of $f^{(-k+1)}$ in (9) yields

$$\sigma_{x_1}[0]=0,\ \text{i.e.,}\ \sigma[0]=g(t,x_2,x_3,\ldots,x_M)\equiv g, \tag{10}$$

$$\sigma[k]=(\partial_x^{-1}K'-\partial_t)\sigma[k-1]=(\partial_x^{-1}K'-\partial_t)^2\sigma[k-2]=\cdots=(\partial_x^{-1}K'-\partial_t)^k g, \tag{11}$$

where $\partial_x^{-1}=\int^x dx$ is an indefinite integral operator. Because of the linearity of the symmetry definition equation (6) we get a generalized formal solution of (6)

$$\sigma=\sum_{k=0}^{\infty}\sigma_n(f_n,g_n)=\sum_{k=0}^{\infty}\sum_{k=0}^{\infty}f_n^{(-k)}(\partial_x^{-1}K'-\partial_t)^k g_n \tag{12}$$

with infinitely many arbitrary functions $f_n$ and $g_n$.

It is interesting that for the KP equation

$$u_{tx}=(6uu_x-u_{xxx})_x-3u_{yy}, \tag{13}$$

if $g_n(y,t)$ is fixed as

$$g_n=\frac{1}{2n!3^{n+1}}y^n, \tag{14}$$

we reobtain the truncated generalized symmetries of the KP equation given in Ref. [9]:

$$\sigma_n(f)=\frac{1}{2n!3^{n+1}}\sum_{k=0}^{n+1}f^{(n+1-k)}(-\partial_x^3+6\partial_x u-3\partial_x^{-1}\partial_y^2-\partial_t)^k y^n,\quad n=0,1,2,\ldots,\sigma_m(f)=0\quad(m<0), \tag{15}$$

where arbitrary function $f$ has been rewritten as $f^{(n+1)}$. The details about the symmetries and algebras of the KP equation can be found in Ref. [9].

*Generalized symmetries and algebras of the 3D Toda theory.*— Based on the general discussions above, we get two sets of generalized symmetries of the 3D Toda equation (3):

$$\sigma_n(f)=\sum_{k=0}^{\infty}f^{-k}(Z)(-\partial-\bar{\partial}^{-1}\phi_{Z\bar{Z}}\partial_s^2)^k g_n(Z,s)\quad(n=0,1,2,\ldots), \tag{16}$$

$$\bar{\sigma}_n(\bar{f})=\sum_{k=0}^{\infty}\bar{f}^{-k}(\bar{Z})(-\bar{\partial}-\partial^{-1}\phi_{Z\bar{Z}}\partial_s^2)^k\bar{g}_n(\bar{Z},s)\quad(n=0,1,2,\ldots), \tag{17}$$

where $\bar{\partial}^{-1}$ and $\partial^{-1}$ are indefinite integral operators with respect to $\bar{Z}$ and $Z$, respectively. In this paper, we focus our attention on the truncated symmetries of (16) and (17) only. Similar to the KP case, after finishing some tedious calculations, we find that if $g_n$ and $\bar{g}_n$ are fixed as

$$g_n=\bar{g}_n=\frac{1}{n!}A_n g^n\quad(n=0,1,2,\ldots) \tag{18}$$

with a constant $A_n$, the formal series (16) and (17) are truncated to

$$\sigma_0(f)=A_0\int^Z f(Z')dZ',$$
$$\sigma_n(f)=\frac{A_n}{n!}\sum_{k=0}^{n-1}f^{n-1-k}(-\partial-\bar{\partial}^{-1}\phi_{Z\bar{Z}}\partial_s^2)^k s^n\quad(n=1,2,\ldots), \tag{19}$$

and $\bar{\sigma}_n(\bar{f})$ can be obtained from $\sigma_n(f)$ after replacing $(f,Z)$ by $(\bar{f},\bar{Z})$, where arbitrary function $f$ has been redefined as $f^{(n-1)}$. The first few $\sigma_n(f)$ read [$\dot{f}=(\partial/\partial Z)f$]

$$\sigma_0(f)=A_0\int^Z f(Z')dZ',\quad \sigma_1(f)=A_1 f(Z)s, \tag{20}$$
$$\sigma_2(f)=-A_2 f(Z)\phi_Z+\tfrac{1}{2}A_2\dot{f}s^2,$$

$$\sigma_3(f)=2A_3 f\bar{\partial}^{-1}\phi_{Z\bar{Z}}\phi_{Zs}-A_3\dot{f}s\phi_Z+\tfrac{1}{6}A_3\ddot{f}s^3, \tag{21}$$

$$\sigma_4(f)=3A_4 f\bar{\partial}^{-1}\phi_{Z\bar{Z}}(\phi_{Z\bar{Z}}-\phi_{Zs}^2)+2A_4\dot{f}(s\bar{\partial}^{-1}\phi_{Z\bar{Z}}\phi_{Zs}+\tfrac{1}{4}\phi_Z^2)-\tfrac{1}{2}A_4\ddot{f}s^2\phi_Z+\tfrac{1}{24}A_4\dddot{f}s^4. \tag{22}$$

After selecting constants $A_0=\frac{1}{2}C^2$, $A_1=\frac{1}{2}C$, $A_2=1$, $A_3=C^{-1}$, $A_4=2C^{-2}$,... ($C$=const), the detailed calculations show us that these symmetries constitute a generalized Lie algebra

$$[\sigma_m(f_1),\sigma_n(f_2)]=\sigma_{m+n-2}[(n-1)\dot{f}_1f_2-(m-1)\dot{f}_2f_1]\,,\quad n,m\geq 0,\quad \sigma_m(f)=0\quad (m<0) \tag{23}$$

under the Lie product

$$[A,B]=A'B-B'A\equiv\frac{\partial}{\partial\epsilon}[A(u+\epsilon B)-B(u+\epsilon A)]|_{\epsilon=0}\,. \tag{24}$$

As in the KP case [9], to get the truncated symmetries (19), $\sigma_n(f)$, we can use $\bar{\partial}^{-1}F_{\bar{Z}}=F$ because we have fixed the integral functions $g_n(f)$ for all $n$. However, to verify the generalized Lie algebras (23) constituted by these symmetries, we have to use the identity

$$\bar{\partial}^{-1}F(s,Z,\bar{Z},\phi_Z,\phi_{\bar{Z}},\ldots)=\bar{\partial}^{-1}(F+0)=\bar{\partial}^{-1}F+h(Z,s) \tag{25}$$

and fix the integration function $h(Z,s)$ such that the right-hand side of Eq. (23) is really a symmetry and coincides with that given by Eq. (19). That is because the Lie product (24) between two known symmetries will yield a unique new symmetry [19] while we have not yet fixed the integral function of the new symmetry. The deeper reason comes from the fact that we cannot define an inverse operator of a partial differential operator when they act on a function with *arbitrary* boundary conditions. It is interesting that the subalgebra constituted by $\sigma_n(f)$ is isomorphic to the symmetry algebra of the KP equation.

Now we would like to list some interesting subalgebras of (23) instead of tedious concrete verification of (23):

(i) It is interesting that from the general expressions of (19), the "time $Z$" independent symmetries

$$\sigma_n\equiv\sigma_n(1)=\frac{A_n}{n!}(-\partial-\bar{\partial}^{-1}\phi_{Z\bar{Z}}\partial_s^2)^{n-1}s^n \tag{26}$$

constitute a *commuting algebra:*

$$[\sigma_n,\sigma_m]=[\bar{\sigma}_m,\bar{\sigma}_n]=[\bar{\sigma}_m,\sigma_n]=0\,. \tag{27}$$

(ii) Starting from the general expression (19), we get the so-called master symmetries [15] of degree $k$ for the 2D Toda equation:

$$\tau_{n,k}=\frac{A_n}{n!}(-\partial-\bar{\partial}^{-1}\phi_{Z\bar{Z}}\partial_s^2)^{n-k-1}s^n\,,\quad k=1,2,\ldots,n-1\,. \tag{28}$$

Especially the master symmetries of degree 1 constitute the Virasoro algebra I:

$$[\tau_{n,1},\tau_{m,1}]=(m-n)\tau_{m+n-2,1}\,, \tag{29}$$

or equivalently

$$[\sigma_n(t),\sigma_m(t)]=(m-n)\sigma_{m+n-2}(t)\quad(m,n=0,1,2,\ldots)\,. \tag{30}$$

(iii) If we take $m=n=2$, $f=\exp(rZ/\alpha)$ ($r=0,\pm 1,\pm 2,\ldots$), and $\alpha$=const, we get the Virasoro algebra II, $\sigma^r\equiv\sigma_2(\exp rZ/\alpha)$:

$$[\sigma^r,\sigma^s]=\frac{1}{\alpha}(r-s)\sigma^{r+s}\quad(r,s=0,\pm 1,\pm 2,\ldots)\,. \tag{31}$$

(iv) When we take $m=n=2$, $f=(1/\alpha)t^r$ ($r=0,\pm 1,\pm 2,\ldots$), and $\alpha$=const, the Virasoro algebra III is obtained immediately $[\sigma^r\equiv\sigma_2(t^r/\alpha)]$:

$$[\sigma^r,\sigma^s]=\frac{1}{\alpha}(r-s)\sigma^{r+s-1}\quad(r,s=0,\pm 1,\pm 2,\ldots)\,. \tag{32}$$

(v) From Eq. (23), we know that if we restrict $m\geq 2$, then $\sigma_m(f)$ constitute a *generalized $w_\infty$ algebra:*

$$[\sigma_m(f_1),\sigma_n(f_2)]=\sigma_{m+n-2}[(n-1)\dot{f}_1f_2-(m-1)\dot{f}_2f_1]\,,\quad m,n\geq 2\,. \tag{33}$$

(vi) In algebra (33) for $m\geq 2$, taking $f=\exp rZ$ ($r=0,\pm 1,\pm 2,\ldots$) leads to the standard $w_\infty$ algebra ($w_\infty$ type I algebra) with $\sigma_m^r\equiv\sigma_m(\exp rZ)$:

$$[\sigma_m^r,\sigma_n^s]=[(n-1)r-(m-1)s]\sigma_{m+n-2}^{r+s}\,,\quad m,n\geq 2\,. \tag{34}$$

It is known that $\sigma_m^r$ is a generator of conformal spin $m$ [2,7,20]. In other types of representations of $w_\infty$ algebra, obtaining the higher spin generators is very difficult [7] while it is quite easy to get the generators for any high conformal spin in our symmetry representation (23) with $f=\exp rZ$.

(vii) The first type of $w_\infty$ algebra (34) is a generalization of the Virasoro algebra II. The second type of $w_\infty$ algebra ($w_\infty$ type II algebra) is a generalization of the Virasoro algebras I and III:

$$[\sigma_m^r,\sigma_n^s]=\frac{1}{\alpha p}[(n-1)r-(m-1)s]\sigma_{m+n-2}^{r+s-p}\,, \tag{35}$$

where $\alpha$=const, $p$ is a fixed integer, and $\sigma_m^r\equiv\sigma_m[(1/\alpha)\times Z^{r/p}]$. The special cases ($r=s=p,\alpha=1$) and ($m=n=2p=2$) are just the Virasoro algebras I and III. To our knowledge no one has studied such types of algebras.

(viii) $w_\infty$ type III algebra is a much stranger $w_\infty$ algebra which is the generalization of the Virasoro algebras II and III $[\sigma_i^r\equiv\sigma_2(Z^{r/p}\exp iZ)]$:

$$[\sigma_i^r,\sigma_j^s]=\frac{1}{\alpha p}(r-s)\sigma_{i+j}^{r+s-p}+\frac{1}{\alpha}(i-j)\sigma_{i+j}^{r+s}\,. \tag{36}$$

The algebra will reduce back to the Virasoro algebras II and III for ($r=s=0$) and ($i=j=0,p=1$), respectively.

In this Letter, two sets of generalized symmetries of the 3D Toda field equation are obtained by using a simple method which yields a simple formula for a general type of high-dimensional nonlinear PDEs. Every set of symmetries constitutes a generalized $w_\infty$ type algebra which is isomorphic to that of the KP equation. Eight types of interesting subalgebras are also discussed. There are three types of Virasoro subalgebras and three types of $w_\infty$ subalgebras. $w_\infty$ type I algebra is a standard one where the generators for arbitrary conformal spin are given.

The other two $w_\infty$ type algebras are less studied in the literature.

The method presented in this paper can easily be extended to other types of linear and nonlinear high-dimensional physics problems which can be described by some linear or nonlinear PDEs. For a linear physics problem, we can directly use the method given here to obtain infinitely many special solutions, say, Eq. (12) for (6) and the fixed function $u$ for the concrete physics problem. For a nonlinear problem, we can get infinitely many symmetries at first, and then using these symmetries to get the infinitely many group invariant solutions of the original nonlinear problem simply by setting $\sigma=0$, where $\sigma$ is an arbitrary symmetry [21].

It is important that in higher-dimensional cases, for some quite general sets of equations, there do exist infinitely many symmetries irrespective of their integrability. From Eq. (16), we have seen that the generalized equation (5), whether it is integrable or not, has an infinite number of symmetries given by Eq. (12) with two *arbitrary* functions. That is to say, in high dimensions, an integrable model has an infinite number of symmetries, but the inverse is not always true. Tamizhmani, Ramani, and Gramaticos [22] have given a concrete example which is a nonintegrable model that has four sets of infinitely many *truncated* symmetries $X_5$, $X_6$, $X_7$, and $X_8$.

From the discussions above we know that the generalized symmetries can be obtained from some different integrable models such as the KP, IDLWE (intergro-differential linear equation), and 3D Toda theory. Naturally, various interesting open questions arise: (1) One has known that the 4D self-dual gravity can be formulated as the sl($\infty$) Toda theory [2,6], so it is worth establishing what the corresponding theory in 4D self-dual gravity is. (2) In the introduction we pointed out that the $w_\infty$ algebra plays an important role in various physical fields, like string theory, $w$ gravity, and membrane theory, then can the generalized symmetry algebra as in the KP and Toda theory be found in other theories, and how can one construct the corresponding string theory, membrane theory, and gravity theory which have the generalized $w_\infty$ symmetry? (3) The quantum $w_\infty$ algebra has also been obtained [12,13]. What is the quantum version corresponding to the generalized $w_\infty$ symmetry algebra given here? (4) Super $w_\infty$ algebra for the super KP equation has been given. How can one get the super extension of the generalized $w_\infty$ algebra? (5) What is the relation between the integrability and truncated condition? (6) Finally, for 3D Toda theory itself there are also many unsolved problems, e.g., giving out the full symmetry algebra for the 3D Toda equation (3) is still a difficult task. There may exist other types of symmetries for the 3D Toda equation (3). For instance, the symmetry, $\sigma=\phi_s$, which corresponds to the $s$-translation invariance is not included in $\sigma_n(f)$ and $\bar{\sigma}_n(\bar{f})$. We would like to discuss these problems in future studies.

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