Exact Calculation of the Ground-State Dynamical Spin Correlation Function of a $S = 1/2$ Antiferromagnetic Heisenberg Chain with Free Spinons

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We calculate the exact dynamical magnetic structure factor $S(Q,E)$ in the ground state of a one-dimensional $S = 1/2$ antiferromagnet with gapless free $S = 1/2$ spinon excitations, the Haldane-Shastry model with inverse-square exchange, which is in the same low-energy universality class as Bethe's nearest-neighbor exchange model. Only two-spinon excited states contribute, and $S(Q,E)$ is found to be a very simple integral over these states.

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The integrable "Haldane-Shastry" model (HSM) [1,2] is a variant of the $S = 1/2$ Heisenberg chain, with exchange inversely proportional to the square of the distance between spins, in contrast to the nearest-neighbor exchange of the Bethe ansatz model (BAM) [3]. The low-energy properties of the antiferromagnetic HSM are in the same universality class as those of the BAM: the elementary excitations are $S = 1/2$ objects [4] ("spinons") obeying *semion* (half-fractional) statistics intermediate between bosons and fermions [5], and the low-energy fixed point is described by the $k = 1$ SU(2) Wess-Zumino-Witten model, a $c = 1$ conformal field theory [1] . A generic model in this universality class has marginally irrelevant interactions that renormalize to zero at the infrared fixed point; this is seen in the BAM. The special feature of the HSM is that such interactions are absent at *all* energy scales [6]: It is in a very real sense the model in which the spinons form an "ideal $S = 1/2$ semion gas," and may be regarded as the fundamental model for gapless half-integral spin antiferromagnetic chains.

While the correlation functions of the BAM have not yet been obtained, the static (equal time) two-point antiferromagnetic ground-state spin correlations of the HSM are already known [1,2]. In this Letter, we extend this result to the full *dynamical* spin correlation function:

$$\langle 0|S_m^a(t)S_n^b(t')|0\rangle = \tfrac{1}{4}\delta^{ab}(-1)^{m-n}C(m-n,t-t')\,, \quad (1)$$

where our expression for $C(x,t)$ is remarkably simple, and directly related to the spinon spectrum. We will first present the result, then describe the derivation.

The HSM Hamiltonian is

$$H = J\sum_{m<n}[d(m-n)]^{-2}\mathbf{S}_m\cdot\mathbf{S}_n, \quad (2)$$

where $d(n) = (N/\pi)\sin(\pi n/N) \to n$ as the number of sites $N\to\infty$. The HSM spinon dispersion relation is

$$\epsilon(q) = (v/\pi)\left[(\pi/2)^2 - q^2\right], \quad v = \pi J/2, \quad (3)$$

which is restricted to *half* the Brillouin zone ($|q| < \pi/2$); v is the low-energy spinon velocity. For $N\to\infty$, a state with $N_{\rm sp}$ spinons has energy $\sum_i \epsilon(q_i)$ and crystal momentum $\exp(iK) = (-1)^M\exp(i\sum_i q_i)$, where $M = (N-N_{\rm sp})/2$ must be integral.

Our concise expression for $C(x,t)$ is

$$\int d\mathbf{M}\,\exp\left(\tfrac{1}{4}i\pi{\rm Tr}\{x\mathbf{M}\mathbf{M}_0 - \tfrac{1}{2}vt[\mathbf{1}-(\mathbf{M}\mathbf{M}_0)^2]\}\right). \quad (4)$$

The integral is over the manifold of 4×4 traceless Hermitian unitary matrices $\mathbf{M} = \mathbf{M}^{-1} = (\hat{\mathbf{\Omega}}_L\cdot\boldsymbol{\sigma}_L)(\hat{\mathbf{\Omega}}_R\cdot\boldsymbol{\sigma}_R)$, where $\{\boldsymbol{\sigma}_L^a\}$ and $\{\boldsymbol{\sigma}_R^a\}$ are two independent sets of 4×4 Hermitian generators that individually obey the same algebra as the Pauli matrices, and which commute with each other; $\hat{\mathbf{\Omega}}_L$ and $\hat{\mathbf{\Omega}}_R$ are real unit 3-vectors. This manifold is isomorphic to the product of two spherical surfaces $S_L^2\times S_R^2$. The time-reversal operator for Pauli matrices is $\boldsymbol{\tau} = i\boldsymbol{\sigma}^2$, so $(\boldsymbol{\sigma}^a)^* = \boldsymbol{\tau}\boldsymbol{\sigma}^a\boldsymbol{\tau}$, and $\boldsymbol{\tau}^2 = -\mathbf{1}$; $\mathbf{M}$ obeys the reality condition $\mathbf{M}^* = (\boldsymbol{\tau}_L\boldsymbol{\tau}_R)\mathbf{M}(\boldsymbol{\tau}_L\boldsymbol{\tau}_R)$ where $(\boldsymbol{\tau}_L\boldsymbol{\tau}_R)^2 = \mathbf{1}$. $\mathbf{M}_0 = (\hat{\mathbf{z}}\cdot\boldsymbol{\sigma}_L)(\hat{\mathbf{z}}\cdot\boldsymbol{\sigma}_R)$ is a point on the manifold. ${\rm Tr}[\mathbf{M}\mathbf{M}_0] = 4\Omega_L^z\Omega_R^z$ and ${\rm Tr}[\mathbf{1}-(\mathbf{M}\mathbf{M}_0)^2] = 8[(\Omega_L^z)^2 + (\Omega_R^z)^2 - 2(\Omega_L^z\Omega_R^z)^2]$.

The invariant measure for the integral is the product of rotationally invariant measures on $S_L^2\times S_R^2$, and the normalization is fixed so $C(0,0) = 1$. Writing $\Omega_{L(R)}^z \equiv \lambda_{1(2)}$, this gives $C(x,t)$ as

$$\tfrac{1}{4}\int_{-1}^{1}d\lambda_1\int_{-1}^{1}d\lambda_2\,e^{iQx-Et}, \quad (5)$$

where $Q = \pi\lambda_1\lambda_2$ and $E = (\pi v/2)(\lambda_1^2+\lambda_2^2-2\lambda_1^2\lambda_2^2)$. Alternatively, if $\pi\lambda_1\lambda_2 = q_1+q_2$, and $(q_1-q_2)^2 = \pi^2(1-\lambda_1^2)(1-\lambda_2^2)$, so $Q = q_1+q_2$ and $E = \epsilon(q_1)+\epsilon(q_2)$, $C(x,t)$ is also given by

$$\frac{v}{(2\pi)^2}\int_{-\pi/2}^{\pi/2}dq_1\int_{-\pi/2}^{\pi/2}dq_2\,\frac{|q_1-q_2|}{[\epsilon(q_1)\epsilon(q_2)]^{1/2}}e^{i(Qx-Et)}. \quad (6)$$

Note that the $S = 1$ state $S_n^a|0\rangle$ is *completely* expressible in terms of eigenstates of the HSM with *only two* parallel-spin spinons carrying momenta q_1 and q_2. Thus

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(6) *explicitly expresses the correlation function in terms of the physical excitations*; this formula is the principal result reported here.

$\langle 0|S_m^a(t)S_n^b(t')|0\rangle$ can be expressed in terms of the *dynamical structure factor* $S(Q,E)$ as

$$\frac{\delta^{ab}}{2\pi}\int_0^{2\pi} dQ \int_0^{\infty} dE\, S(Q,E)e^{i[Q(m-n)-E(t-t')]}, \qquad (7)$$

where $S(Q,E)$ is explicitly given by

$$\tfrac{1}{4}\theta(E_2(Q)-E)\prod_{\nu=\pm}\left[\frac{\theta(E-E_{1\nu}(Q))}{[E-E_{1\nu}(Q)]^{1/2}}\right], \qquad (8)$$

where $E_{1-}(Q)=(v/\pi)Q(\pi-Q)$, $E_{1+}(Q)=(v/\pi)(Q-\pi)(2\pi-Q)$, and $E_2(Q)=(v/2\pi)Q(2\pi-Q)$. Integration over E gives the previously known expression for the static structure factor: $S(Q)=\frac{1}{4}\ln(|1-Q/\pi|^{-1})$.

Our calculation is based on the identification of $C(x,t)$ as proportional to the *bosonic* single-particle ground-state correlation function $\langle 0|\Psi^\dagger(x',t')\Psi(0,0)|0\rangle$ (with $x'\propto x$, $t'\propto t$) for the Calogero-Sutherland model (CSM) [7] of a spinless 1D Galilean-invariant gas with interactions $(\hbar^2/m)[\lambda(\lambda-1)/(x_i-x_j)^2]$, at coupling constant $\beta=2\lambda=4$, the coupling at which it is related to the symplectic random matrix ensemble [8].

Though different in its technical details, a recent calculation by Simons, Lee, and Altshuler (SLA) [9] of $\langle 0|\rho(x',t')\rho(0,0)|0\rangle$ for the $\beta=4$ CSM suggested our calculation. Our result shows that the *hole* excitation in this model fractionalizes into two semions [as in the essentially similar bosonic Laughlin fractional quantum Hall effect (FQHE) state at Landau level filling $\nu=1/2$]. Our interpretation of the SLA expression [9] is that it shows that the *particle* excitation remains elementary (the FQHE analog is a particle added *outside* the Laughlin droplet); inspection of their formula shows it to be the convolution of our result for the fractionalized hole contribution with a factor representing an additional particle excitation outside the ground state condensate.

We first note that (for finite N) the HSM exhibits not just SU(2) symmetry, with an associated sl_2 Lie algebra generated by $\mathbf{J}_0=\sum_m \mathbf{S}_m$, but a remarkable additional *quantum symmetry*, the *Yangian* $\mathrm{Y}(\mathrm{sl}_2)$ symmetry algebra with the additional generator [10]

$$\mathbf{J}_1=\frac{h}{2}\sum_{m<n}\cot\left(\frac{\pi(m-n)}{N}\right)\mathbf{S}_m\times\mathbf{S}_n, \qquad (9)$$

where $[H,J_m^a]=J_m^a|0\rangle=0$, $m=0,1$. Here h is the "quantum parameter" of $\mathrm{Y}(\mathrm{sl}_2)$; this is a scale parameter, conventionally rescaled to 1. However, in the limit $N\to\infty$, hN must be held constant, so $h\to 0$. The symmetry algebra then is the "classical" infinite-dimensional Lie algebra $\widehat{\mathrm{sl}_2}_+$, which is the algebra of non-negative modes of a Kac-Moody algebra: $[J_m^a,J_n^b]=i\epsilon^{abc}J_{m+n}^c$, $m,n\geq 0$, with $J_m^a|0\rangle=0$, $m\geq 0$.

HSM eigenstates are organized into Yangian "super-multiplets" [1,10] [of many degenerate SU(2) multiplets] each containing a single Yangian highest weight state (YHWS) which is the state of highest J_0^z in the multiplet. The set of YHWS span a subspace of states that can be physically characterized as the "fully polarized spinon gas" (FPSG) states [10], where all spinons have maximally polarized spins. The wave functions for FPSG states can be conveniently expressed in terms of M complex coordinates Z_i [with $(Z_i)^N=1$] of lattice sites (on the unit circle) with reversed spins:

$$\Psi(\{Z_i\})=\Phi(\{Z_i\})\prod_{i<j}(Z_i-Z_j)^2\prod_i Z_i, \qquad (10)$$

where $\Phi(\{Z_i\})$ is a symmetric polynomial of degree $N_{\rm sp}$ in each variable; the constraint $M=(N-N_{\rm sp})/2$ ensures that the Fourier transform of the wave function has *compact support* within a *single* Brillouin zone. Diagonalization of the HSM Hamiltonian within the FPSG subspace gives the YHWS. The other eigenstates are obtained from the YHWS by repeated action of the Yangian generators, and the multiplets are classified as irreducible representations of $\mathrm{Y}(\mathrm{sl}_2)$ [10,11]. The structure of these multiplets is reminiscent of that of the ideal $S=1/2$ Fermi gas, where there is an independent spin degree of freedom for each orbital; $\mathrm{Y}(\mathrm{sl}_2)$ multiplets are also isomorphic to direct products of independent sl_2 representations, but in contrast to the fermion case, the "orbital" or "Fock space" structure *changes* with particle number. These degeneracies make it appropriate to identify the excitations of the HSM as forming an "ideal spinon gas" [6,10].

The operator S_m^+ acting on a FPSG state removes a down-spin coordinate, and preserves the FPSG character of the state. The wave function describing $S_m^+|0\rangle$ is

$$\prod_i(Z_i-z_m)^2\prod_{i<j}(Z_i-Z_j)^2\prod_i Z_i. \qquad (11)$$

The fact that the initial polynomial factor is degree 2 in each coordinate Z_i immediately shows that this state is composed only of states with *two* spinon excitations.

The polynomial character of the FPSG wave functions implies that when matrix elements of the Hamiltonian between such states are calculated, sums over discrete lattice sites on the unit circle can be replaced [1,6] by integrals of a continuous coordinate on the unit circle (the compact support of the Fourier transforms of these polynomials within the first Brillouin zone in reciprocal space means that all "umklapp" corrections vanish). Thus calculations for the HSM involving only FPSG states will give *identical* results to analogous calculations for a continuous model, the $\lambda=2$ CSM on a circle. Then $C(x,t)\propto\langle 0|\Psi^\dagger(x',t')\Psi(0,0')|0\rangle$ where x' and t' are space and time coordinates suitably scaled in terms of CSM units.

Because of the special form of the HSM and CSM ground states, $C(x,t)$ can also be written

$$\langle 0|\prod_i (Z_i - z')^2 e^{-i(H-E_0)t'} \prod_i (Z_i - z)^2|0\rangle. \qquad (12)$$

We note that the method of our calculation can easily be extended to obtain $\langle 0|\prod_j \Psi^\dagger(x_j,t)\prod_j \Psi(x'_j,t')|0\rangle$ where n particles are removed at various places at one time t' and replaced at different places at a different time t. (The final integration over a manifold of $4n$-dimensional matrices will be more complicated, however.)

In the thermodynamic limit, the correlations of the CSM for particles confined in a weak harmonic potential that is rescaled to maintain a constant particle density in some region become locally equivalent to those of the same density CSM model on a circle. We use this Gaussian formulation, since in this case [12], the dynamics of the CSM particles become *identical* to the dynamics of the doubly degenerate eigenvalues of a Hermitian matrix with matrix elements that are harmonic oscillators constrained so the matrix has Kramers degeneracy. This is the *Gaussian Hermitian matrix model* [12] with symplectic symmetry.

The action for the matrix model is

$$S = \tfrac{1}{2}\int dt\, \mathrm{Tr}[(\dot{\mathbf{Q}})^2 - \omega^2(\mathbf{Q})^2], \qquad (13)$$

where $\mathbf{Q}$ is a $2N \times 2N$ Hermitian matrix

$$\mathbf{Q} = \mathbf{S}\otimes\mathbf{1} + \sum_a \mathbf{A}^a \otimes \boldsymbol{\sigma}^a. \qquad (14)$$

Here $\mathbf{S}$ is a real symmetric $N\times N$ matrix and $\mathbf{A}^a$ are three independent imaginary antisymmetric matrices, making $2N^2 - N$ independent coordinates in total. Then $\mathbf{Q}^* = \mathbf{T}\mathbf{Q}\mathbf{T}^{-1}$, where $\mathbf{T} = \mathbf{1}\otimes\boldsymbol{\tau}$, and $\mathbf{T}^2 = -\mathbf{1}$, which ensures Kramers degeneracy. The limit $N \to \infty$ with $\omega \propto 1/N$ is taken at the end of the calculation.

The SLA calculation [9] was originally a quenched random matrix ensemble average, but can be reformulated [13] in terms of a dynamical Gaussian Hermitian matrix model, when it is equivalent to a calculation of

$$\langle 0|T_t \prod_{\alpha=\pm} \mathrm{Tr}\{[\mathbf{Q}(\alpha t) - \alpha x \pm i0^+]^{-1}\}|0\rangle\,,$$

which is a more complex calculation than the one we describe here.

In (12), if the $\{Z_i\}$ are interpreted as the doubly degenerate eigenvalues of a unitary matrix $\mathbf{U}$ with Kramers degeneracy belonging to Dyson's circular ensemble with symplectic symmetry [8], $\prod_i (Z_i - z)^2 \equiv \det[\mathbf{U} - z]$. Invoking the large-$N$ local equivalence of the eigenvalue distributions of the circular unitary and Gaussian Hermitian ensembles [8], we identify the Gaussian matrix-model quantity we need as

$$\langle 0|T_t \prod_{\alpha=\pm} \det[\mathbf{Q}(\alpha t) - \alpha x]|0\rangle, \qquad (15)$$

where the space and time coordinates have been rescaled in terms of matrix model units, and will be held fixed as $N \to \infty$.

The fundamental matrix-model correlation function is

$$\langle 0|T_t \mathrm{Tr}[\mathbf{A}\mathbf{Q}(t)]\mathrm{Tr}[\mathbf{B}\mathbf{Q}(t')]|0\rangle = G(\mathbf{A},\mathbf{B})g(t-t'), \quad (16)$$

where $g(t) = (\hbar/2\omega)\exp(i\omega|t|)$, $\mathbf{A}$ and $\mathbf{B}$ are c-number matrices, and $G(\mathbf{A},\mathbf{B})$ is

$$\tfrac{1}{2}\sum_{ij\sigma\sigma'} \left(A_{ij}^{\sigma\sigma'} B_{ji}^{\sigma'\sigma} + A_{ij}^{\sigma\sigma} B_{ij}^{\sigma'\sigma'} - A_{ij}^{\sigma\sigma'} B_{ij}^{\sigma'\sigma} \right). \qquad (17)$$

The determinant is given by a Gaussian Grassmann-number integral:

$$\det[\mathbf{Q}(\alpha t)] = \int \prod_{i\sigma} d\psi_{i\sigma}^*(\alpha) d\psi_{i\sigma}(\alpha) \exp \mathrm{Tr}[\mathbf{A}(\alpha)\mathbf{Q}(\alpha t)],$$

where $A_{ij}^{\sigma\sigma'}(\alpha) = \psi_{i\sigma}(\alpha)\psi_{j\sigma'}^*(\alpha)$.

We now evaluate the harmonic oscillator correlation function, using the standard harmonic oscillator result $\langle 0|T_t \exp A_1 \exp A_2|0\rangle = \exp[(1/2)\sum_{ij}\langle 0|T_t A_i A_j|0\rangle]$ where A_i are linear functions of the coordinates, and obtain

$$\prod_{i\sigma\alpha} \int d\psi_{i\sigma}^*(\alpha) d\psi_{i\sigma}(\alpha) \exp S[\{\Psi_{i\sigma}^*(\alpha), \Psi_{i\sigma}(\alpha)\}]; \qquad (18)$$

here $S = xS_1 + (\hbar/8\omega)S_2$ where $S_1 = \sum_\alpha \alpha\rho_{\alpha\alpha}$ and $S_2 = \sum_{\alpha\beta}[g_{\alpha\beta}(t)]^2 G(\mathbf{A}(\alpha), \mathbf{A}(\beta))$, with

$$G(\mathbf{A}(\alpha), \mathbf{A}(\beta)) = -\left(\rho_{\alpha\beta}\rho_{\beta\alpha} + \tfrac{1}{2}\phi_{\alpha\beta}^*\phi_{\beta\alpha}\right), \qquad (19)$$

and $g_{\alpha\beta}(t) = \delta_{\alpha\beta} + (1-\delta_{\alpha\beta})\exp(i\omega|t|)$. Here $\rho_{\alpha\beta} = \sum_{i\sigma} \psi_{i\sigma}^*(\alpha)\psi_{i\sigma}(\beta)$, $\phi_{\alpha\beta} = \sum_{i\sigma\sigma'} \tau_{\sigma\sigma'}\psi_{i\sigma}(\alpha)\psi_{i\sigma'}(\beta)$, and $\phi_{\alpha\beta}^* = \sum_{i\sigma\sigma'} \tau_{\sigma\sigma'}\psi_{i\sigma}^*(\alpha)\psi_{i\sigma'}^*(\beta)$.

We now carry out a bosonic Hubbard-Stratonovitch transformation to decouple S_2; ten independent real integration variables are required, which can be organized into a 2×2 Hermitian matrix $V_{\alpha\beta}$ coupling to $\rho_{\alpha\beta}$ and a 2×2 complex symmetric matrix $\Delta_{\alpha\beta}$ coupling to $\phi_{\alpha\beta}$, with $\Delta_{\alpha\beta}^*$ coupling to $\phi_{\alpha\beta}^*$. These can in turn be organized into a 4×4 matrix $\mathbf{M}(\mathbf{V}, \boldsymbol{\Delta}, \boldsymbol{\Delta}^*)$ where $M_{\alpha 1,\beta 1} = V_{\alpha\beta}$, $M_{\alpha 1,\beta 2} = \Delta_{\alpha\beta}$, $M_{\alpha 2,\beta 1} = \Delta_{\alpha\beta}^*$, and $M_{\alpha 2,\beta 2} = V_{\beta\alpha}$. Provided $V_{\beta\alpha} = (V_{\alpha\beta})^*$ and $\Delta_{\alpha\beta}^* = (\Delta_{\beta\alpha})^*$, this matrix is Hermitian, and obeys the reality condition

$$\mathbf{M}^* = (\mathbf{1}\otimes\boldsymbol{\sigma}^1)\mathbf{M}(\mathbf{1}\otimes\boldsymbol{\sigma}^1). \qquad (20)$$

Such a matrix has the ten-parameter form

$$\mathbf{M} = C^0\mathbf{1} + \sum_{ab} C^{ab}\boldsymbol{\sigma}_L^a\boldsymbol{\sigma}_R^b, \qquad (21)$$

where C^0 is a real number and $\mathbf{C}$ is a real 3×3 matrix. The explicit realization of the generators is $\boldsymbol{\sigma}_L = (\boldsymbol{\sigma}^2\otimes\boldsymbol{\sigma}^2, -\boldsymbol{\sigma}^2\otimes\boldsymbol{\sigma}^1, \mathbf{1}\otimes\boldsymbol{\sigma}^3)$ and $\boldsymbol{\sigma}_R = (\boldsymbol{\sigma}^1\otimes\boldsymbol{\sigma}^3, \boldsymbol{\sigma}^2\otimes\mathbf{1}, \boldsymbol{\sigma}^3\otimes\boldsymbol{\sigma}^3)$. The unitary transformations generated by $\{\boldsymbol{\sigma}_L^a, \boldsymbol{\sigma}_R^a\}$ correspond to a $G = \mathrm{SO}(3)_L\otimes\mathrm{SO}(3)_R$ continuous symmetry group, under the action of which $\mathbf{C} \to \mathbf{O}_L\mathbf{C}\mathbf{O}_R^{-1}$.

We now take the large-N limit with $\omega \propto 1/N$, and $x', t' \sim O(1)$. The Grassmann integrals decouple into a product of N identical integrals leading to a factor

$$\left(\det[\mathbf{M} + iN^{-1}\delta\mathbf{M}]\right)^N , \tag{22}$$

where (dropping scale factors in x' and t')

$$\delta\mathbf{M} = (x' - t'[\mathbf{M}, \mathbf{M}_0])\mathbf{M}_0, \tag{23}$$

$\mathbf{M}_0 = (\boldsymbol{\sigma}^3 \otimes \mathbf{1})$. The correction $\delta\mathbf{M}$ shifts the matrix in the determinant resulting from the Grassmann integrals away from the Hermitian value $\mathbf{M}$ of the Hubbard-Stratonovich fields, but this does not affect the validity or convergence of the Grassmann integral. Using the identity $\det[\mathbf{M}] \equiv \exp(\mathrm{Tr}[\ln\mathbf{M}])$, the value of the integrand for large N, with the Gaussian Hubbard-Stratonovich factor included, becomes $\{\det[\mathbf{M}]\exp(-\mathrm{Tr}[\mathbf{M}^2]/2)\}^N \exp(i\varphi)$ where $\varphi = \mathrm{Tr}\{\mathbf{M}^{-1}(x' - t'[\mathbf{M}, \mathbf{M}_0])\mathbf{M}_0\}$. The factor which is exponentiated to a power N is maximized when $\mathbf{M}^2 = \mathbf{1}$, or $\mathbf{M}^{-1} = \mathbf{M}$, so at the extremal points,

$$\varphi = \mathrm{Tr}\{x'\mathbf{M}\mathbf{M}_0 - t'[\mathbf{1} - (\mathbf{M}\mathbf{M}_0)^2]\}. \tag{24}$$

There are five classes of solutions of the condition $\mathbf{M}^2 = 1$ corresponding to the five possible signatures $s = \pm 4$, ± 2, and 0 of $\mathbf{M}$, all with $\mathrm{Tr}[\mathbf{M}] = s$ and $C^0 = s/4$. For $s = \pm 4$, there are two discrete solutions with $\mathbf{C} = 0$, invariant under the full group G. For $s = \pm 2$, $\mathbf{C} = C^0\mathbf{O}$, where $\mathbf{O}$ is orthogonal, which is invariant under the SO(3) subgroup of G where $\mathbf{O}_L = \mathbf{O}\mathbf{O}_R\mathbf{O}^{-1}$, giving two manifolds of dimension 3. Finally, for $s = 0$, $C^{ab} = \Omega_L^a\Omega_R^b$ where $\hat{\mathbf{\Omega}}_{L(R)}$ are unit 3-vectors; this is invariant under the $\mathrm{O}(2)_L \otimes \mathrm{O}(2)_R$ subgroup of G where $\mathrm{O}_{L(R)}$ is a rotation around $\hat{\mathbf{\Omega}}_{L(R)}$, and forms a manifold of dimension 4.

When the same extremal value occurs on several different manifolds, only manifolds with the largest dimension contribute to the integral as $N \to \infty$. The measure for the integral over the dominant $s = 0$ extremal manifold is just the invariant measure on $S_L^2 \times S_R^2$. To complete the derivation of (4), the space and time variables x' and t' must be rescaled into HSM units. This is accomplished by noting that the two-spinon states span a momentum range of 2π and an energy range of $\pi v/2$.

Finally, we comment on the relation between our HSM result (6) and the BAM correlations. An ansatz similar to the exact HSM result (8) for $S(Q,E)$ has previously been proposed [14] as an *approximation* to the BAM structure factor. The key simplification in the HSM is that only the two-spinon excited states contribute to $S(Q,E)$ at $T = 0$. In the BAM, the numerical studies [14] show that while these states dominate the spectral weight, there is a finite contribution from the $N_{\mathrm{sp}} > 2$ states which, while small, presumably remains finite for any large but finite N_{sp}.

A variant of the HSM with $N = \infty$ and $d(n) = \kappa^{-1}\sinh(\kappa n)$, κ real, has been identified [15] as a family of integrable models that interpolate between the $N = \infty$ HSM ($\kappa = 0$) and the BAM ($\kappa = \infty$). These models also have a $\mathrm{Y(sl_2)}$ quantum symmetry [10] generated by the analog of (9) with $\cot[\pi(m-n)/N]$ replaced by $i\coth[\kappa(m-n)]$; this realization of $\mathrm{Y(sl_2)}$ goes over smoothly into its more familiar realization in the $N = \infty$ BAM limit.

The Yangian quantum parameter h must be rescaled to zero to keep h/κ constant in the HSM limit $\kappa \to 0$ with $N = \infty$. The reverse process by which the "classical" $\widehat{\mathrm{sl}_2}_+$ symmetry at $\kappa = 0$ becomes the $\mathrm{Y(sl_2)}$ "quantum" symmetry when $\kappa > 0$ is a *"quantum deformation."* We speculate that the extension of the $\kappa = 0$ result (6) to the $\kappa > 0$ model [15], and thence to the $\kappa = \infty$ BAM limit, may be achievable as such a "quantum deformation." In this scenario, there is an expression for $C(m,t;\kappa)$ as an expansion in multispinon terms with $N_{\mathrm{sp}} = 2, 4, 6, \ldots, \infty$; when $\kappa = i\pi/N$ the $N_{\mathrm{sp}} > 2$ terms would vanish, giving the finite-N HSM correlations.

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