Spectral Exponents of Enstrophy Cascade in Stationary Two-Dimensional Homogeneous Turbulence

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Direct numerical simulations with up to 4096^2 resolution are performed to address the question of universality of statistical properties of the enstrophy cascade in homogeneous two-dimensional turbulence driven by large-scale Gaussian white-in-time noise. Data with different Reynolds numbers are compared with each other. The energy spectrum is found to be very close to $1/k^3$. It is shown that the primary contribution to the enstrophy transfer function comes from wave-number triads with one small leg and two long ones, corresponding to wave numbers in the inertial range.

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Recently Polyakov [1] suggested to apply the methods of conformal field theory to the problem of two-dimensional turbulence and predicted that the energy spectrum for the enstrophy cascade might have anomalous scaling quite different from the $\log^{-1/3}k/k^3$ law proposed by Kraichnan [2]. Alternatively, Falkovich and Lebedev [3] attempted to construct a rigorous theory out of Kraichnan's idea [4] that enstrophy cascade is analogous to distortion of a passive scalar field (vorticity) by a large-scale straining field. The experimental predictions for the energy spectrum seems to be even more controversial. The majority of works in the field have concentrated on the case of decaying turbulence. The results of those experiments depend strongly on the initial conditions of the decay [5]. In the transient period of decay it is possible to observe the energy spectrum $1/k^3$ [6], but the spectral exponent changes in time. The emergence of strong coherent vortices [7] makes any prediction even more difficult. Stationary two-dimensional turbulence was explored in Ref. [8] (256^2), [9] (512^2), and [10] (1024^2). In [9] and [10] the significant deviation from a $1/k^3$ spectrum was found. The $1/k^3$ law was observed in [8] when all coherent vortices were destroyed by strong infrared hyperviscosity. We report large-scale computer simulations that explore Reynolds number dependences of stationary homogeneous two-dimensional turbulence with the resolution up to 4096^2.

The simulated equation has the form

$$\partial_t\omega+\partial_x\psi\partial_y\omega-\partial_y\psi\partial_x\omega =(-1)^{p_i+1}\nu_i\Delta^{-p_i}\omega+(-1)^{p_u+1}\nu_u\Delta^{p_u}\omega+F\,, \quad (1)$$

where ψ is the stream function, the vorticity $\omega=\Delta\psi$, and the velocity $v_i=\varepsilon_{ij}\partial_j\psi$. The right-hand side of (1) contains a white noise in time Gaussian force which is nonzero only at some characteristic scale k_f:

$$\langle F(\mathbf{k},t)F(\mathbf{k}',t')\rangle\sim\delta(k^2-k_f^2)\delta(\mathbf{k}+\mathbf{k}')\delta(t-t')\,, \quad (2)$$

and two artificial dissipative terms designed to provide an energy sink at large scales and an enstrophy sink at small scales. Enstrophy is defined as half the squared vorticity. We performed two series of simulations: first with normal viscosity $p_u=1$, and then with hyperviscosity $p_u=8$ (parameters are given in Table I). For the energy sink at small k we used a hyperviscosity with $p_i=8$. The pseudospectral parallel code described in [11] was used. The

TABLE I. Enstrophy cascade parameters: N, resolution; k_f, scale of the force; k_i,k_u, infrared and ultraviolet cutoffs; $k_d=(\eta/\nu^3)^{1/6}$; $\mathrm{Re}_u=k_u^2/k_f^2$, Reynolds number; δt, time step; $t_{\mathrm{eddy}}\approx 2\pi/\omega_{\mathrm{rms}}$, large eddies turnover time; $t_\eta\approx(\eta)^{-1/3}$; T_{tot}, total time of integration.

	Normal viscosity				Hyperviscosity			
				Labels				
Parameters	$N1$	$N2$	$N3$	$N4$	$H1$	$H2$	$H3$	$H4$
N	512	1024	2048	4096	512	1024	2048	1024
k_f	4–6	4–6	4–6	4–6	4–6	4–6	4–6	12–14
k_d	28	40	135	214	Not applicable			
k_u	13	16	43	59	200	390	740	365
k_i	1.8	2	1.7	2	1.8	1.5	2	1.8
Re_u	4.7	7.1	51	97	1100	4200	15 200	680
δt	3×10^{-3}	1×10^{-3}	5×10^{-4}	2×10^{-4}	2×10^{-3}	8×10^{-4}	5×10^{-4}	4×10^{-4}
t_{eddy}	11	10.3	8	7	6	5.3	5.7	3.3
t_η	6.3	6.3	5.5	5.5	5	5	4.6	2.4
T_{tot}	300	120	80	30	2000	300	100	100

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parameters were chosen in such a way as to keep the infrared part of the system practically the same for all the simulations except $H4$.

If the energy input rate at the scale of the force k_f is ε, then the enstrophy input rate at the same scale is $\eta = \varepsilon k_f^2$. The conservation of energy and enstrophy leads to $\varepsilon = J_E^i + J_E^u$ and $\eta = J_H^i + J_H^u$, where $J_{E,H}^{i,u}$ are infrared and ultraviolet energy and enstrophy fluxes, respectively. If we define the infrared and ultraviolet cutoffs k_i, k_u as $k_u^2 = J_H^u/J_E^u$, $k_i^2 = J_H^i/J_E^i$, the ratio of the fluxes will be $J_E^u/J_E^i = (k_f^2 - k_i^2)/(k_u^2 - k_f^2)$. There are two natural Reynolds numbers:

$$\mathrm{Re}_u = k_u^2/k_f^2, \quad \mathrm{Re}_i = k_f^2/k_i^2 . \tag{3}$$

It is clear that only the infrared energy J_E^i and ultraviolet enstrophy J_H^u fluxes survive in the limit of infinite Reynolds numbers. We expect that energy spectrum $E(k)$ has the form $E(k) = \eta^{2/3}/k^3 G(k/k_u, \mathrm{Re}_u, \mathrm{Re}_i)$ with possible anomalous dimensions included in the function G. If the viscosity is normal, it is also useful to define the dissipation scale $k_d = (\eta/\nu^3)^{1/6}$ introduced by Kraichnan [2]. Experimental data show that for this case $k_u \sim k_d^{3/4}$, and the Reynolds number in the standard definition k_d^2/k_f^2 is proportional to Re_u.

Energy spectra for all simulations are shown in Fig. 1(a) for normal viscosity and in Fig. 1(b) for hyperviscosity. Plotted as functions of k/k_d for normal viscosity and as functions of k/k_u for hyperviscosity, the data show universality near the ultraviolet cutoff and are more scattered near the force. The larger the Reynolds number, the better the $1/k^3$ law for the energy spectrum is satisfied. The energy spectrum for the normal viscosity simulations has an approximately exponential dissipation range, while the dissipation range for hyperviscosity may be approximated by the formula $E(k) \sim \exp[-\exp(5k/k_u)]$. We define the inertial range as the range of constant enstrophy flux,

$$\begin{aligned} J_H(k) &= \int_{|k'|>k} T_H(k')dk' \\ &= \int_{|k'|>k} T_H(\mathbf{k}',\mathbf{p},\mathbf{q}) d^2k' d^2p\, d^2q , \end{aligned} \tag{4}$$

where

$$\begin{aligned} T_H(\mathbf{k},\mathbf{p},\mathbf{q}) = \varepsilon_{ij} p_i q_j k^2 (\mathbf{p}^2 - \mathbf{q}^2) \delta(\mathbf{k}+\mathbf{p}+\mathbf{q}) \\ \times \psi(\mathbf{k}) \psi(\mathbf{p}) \psi(\mathbf{q}) . \end{aligned} \tag{5}$$

$J_H(k)$ is plotted in Fig. 2(a). As may be seen the range of nearly constant flux starts from $k/k_u \approx 0.7$ for the hyperviscosity runs and from $k/k_d \approx 0.25$ for the normal viscosity runs and continues quite close to k_f. There is no inertial range for the low resolution normal viscosity cases $N1, N2$. The plots for energy spectra in the inertial range are given in Fig. 2(b).

It can be seen from Fig. 2(b) that both the force and the dissipation lead to corrections of $1/k^3$ law. In Figs. 3(a) and 3(b) we plot $\log_{10}[E(k)k^3/\eta^{2/3}]$ as a function of $\log_{10}[\log_{10}(k/k_f)]$ and $\log_{10}[\log_{10}(k_u/k)]$. As may be seen near $k \sim k_u$ the energy spectrum is corrected as $E(k) \sim \eta^{2/3} \log_{10}(k_u/k)^{1/3}/k^3$. Near $k \sim k_f$ the correction is $E(k) \approx C\eta^{2/3} \ln(k/k_f)^{-1/3}/k^3$. This form of the spectrum was predicted by Kraichnan with the Kolmogorov constant $C = 2.626$ [2]. The measured value of $C \approx 1.5$–1.7 is slightly different, but we clearly do not

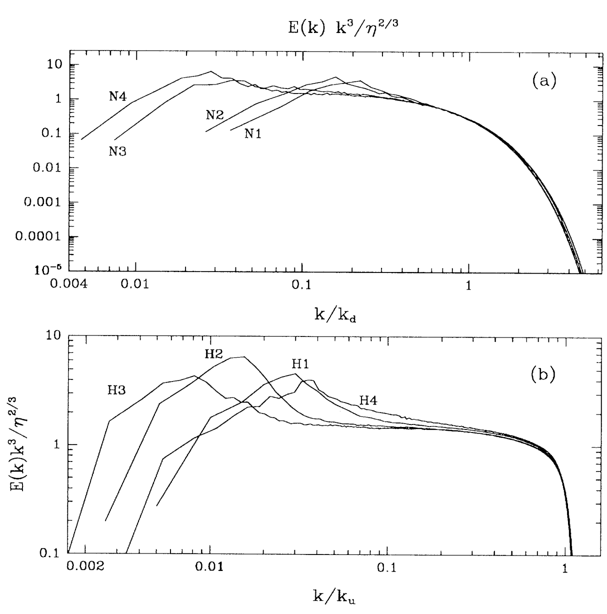

FIG. 1. Log-log plots of compensated energy spectra $E(k)k^3/\eta^{2/3}$: (a) normal viscosity; (b) hyperviscosity.

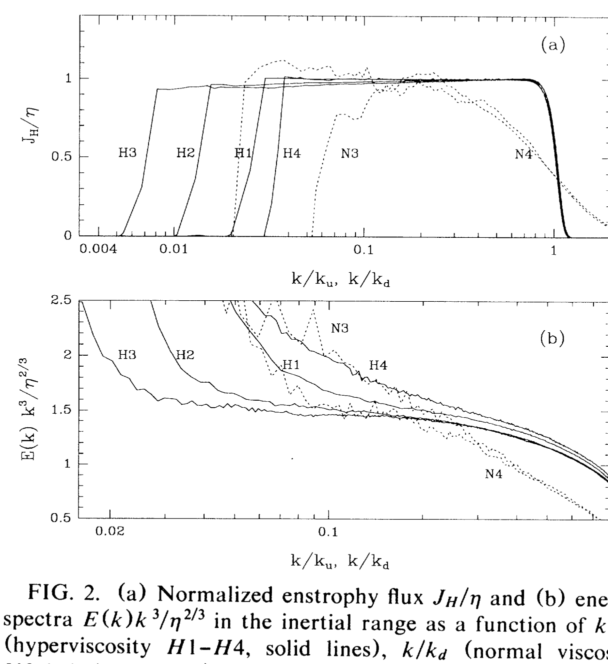

FIG. 2. (a) Normalized enstrophy flux J_H/η and (b) energy spectra $E(k)k^3/\eta^{2/3}$ in the inertial range as a function of k/k_u (hyperviscosity $H1$–$H4$, solid lines), k/k_d (normal viscosity $N3, N4$, dotted lines).

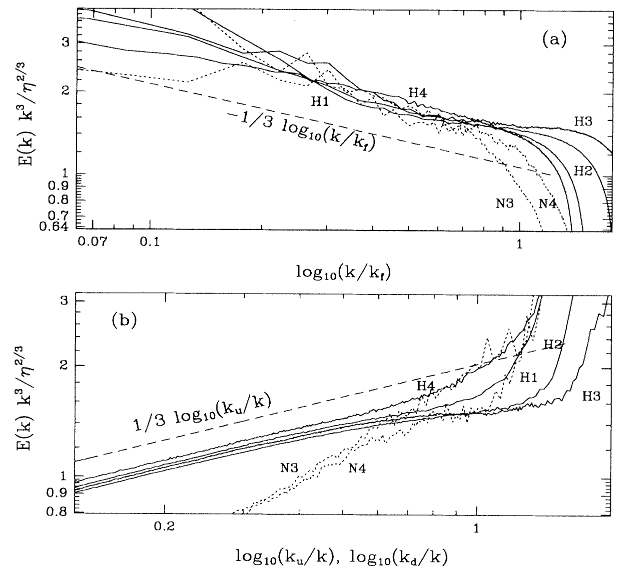

FIG. 3. The log-log plots of energy spectra $E(k)k^3/\eta^{2/3}$ (a) as a function of $\log_{10}(k/k_f)$; (b) as a function of $\log_{10}(k/k_{u,d})$; solid lines, hyperviscosity, dotted lines, normal viscosity.

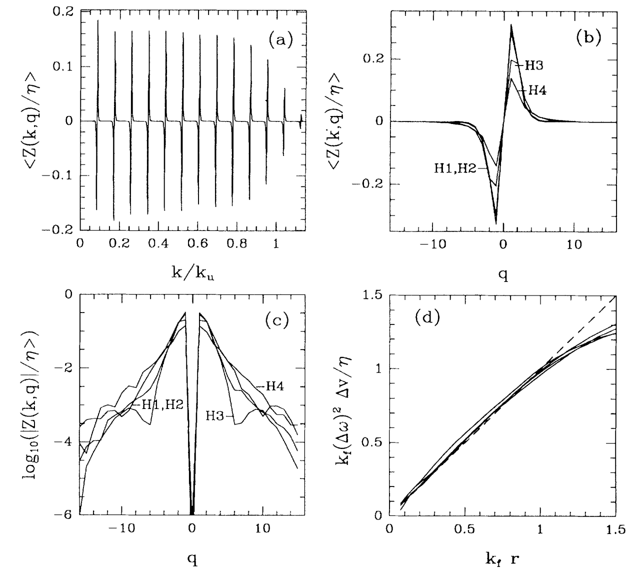

FIG. 4. (a) Enstrophy transfer function $Z(k,q)/\eta$, measured at $k=64,128,\ldots$ as a function of k/k_u for the $H3$ run; (b) one of the spikes of $Z(k,q)$ as a function of q; (c) the same spikes in log scale; (d) $k_f\Delta\omega^2\Delta v/2\eta$, dashed line is the Kolmogorov law for this quantity.

have enough inertial range to find out corrections to the Kraichnan law in the "deep" inertial range. For the normal viscosity case the behavior near k_f is practically the same as in the case of hyperviscosity. However, near the ultraviolet cutoff $k_{d,u}$ the energy spectrum behaves quite differently for the different viscosity types. It is hard to tell whether this near-dissipation range correction will be a universal property of the system, or it is an artifact of the specific form of the hyperviscosity. Although we cannot completely rule out the possibility of anomalous dimensions, for the given setup they should be very small.

The mechanism of the enstrophy transfer is one of the most important characteristics of the cascade. We will demonstrate that enstrophy transfer is very nonlocal in wave-number space. We measured the enstrophy transfer function averaged over angles:

$$Z(k,q)=\int \delta(|k'|-k-q)\delta(|p'|-k)T_H(\mathbf{k}',\mathbf{p}',\mathbf{q}')\times d^2k'd^2p'd^2q' . \quad (6)$$

The simplest way to measure the function of two variables $Z(k,q)$ is to pick a uniform sequence of k and measure the transfer function at $k+q$ coming only from small q. The results of the time averaging for $Z(k,q)$ are plotted in Fig. 4. The centers of the spikes in Fig. 4(a) correspond to the chosen sequence of k, and the spikes themselves give the dependence of $Z(k,q)$ on q, where q is the deviation from the centers of spikes. It was checked that the form of individual spikes in $Z(k,q)$ is practically independent of k, except in the dissipation range. The individual spikes as a function of q are plotted in Figs. 4(b) and 4(c). It may be seen that $Z(k,q)$ falls off very rapidly, practically exponentially. Nearly all the contributions to the transfer function come from q smaller than the scale of the force k_f, i.e., from wave numbers not belonging to the enstrophy cascade inertial range. The function $Z(k,q)$ is an odd function of q. The total transfer $T_H(k)$ (4) is equal to $T_H(k)=\int Z(k,q)dq$, since $T_H(k)=0$ in the inertial range. For a fast decaying $Z(k,q)$ it may be shown that the total enstrophy flux $J_H(k)$ may be represented as

$$J_H(k)\approx \int_{k'<k} dk' \int_{k-k'}^{\infty} Z(k',q)dq . \quad (7)$$

We have confirmed (7) numerically. The fact that $Z(k,q)$ is approximately an odd function of q means that the influence of two large k on a small one is negligible. If $T_H(k,p,q)$ is nonzero in the inertial range, the time reversibility is broken. The result that $T_H(k,p,q)\approx 0$ when all k,p,q lay in the inertial range may confirm the conjecture that turbulence may be time reversible in the inertial range, and the arrow of time is set by wave numbers not belonging to the inertial range himself, the "infrared leakage" of time symmetry according to Polyakov [1]. The similar results for the enstrophy transfer function were reported for smaller resolutions in [12,13].

The direct way to check the consistency of different theories would be to measure higher order vorticity correlators directly in Fourier space. Unfortunately, the fluctuations of these correlation functions are so strong in comparison with their averages that this goal is very hard to accomplish. We tried to measure some higher order

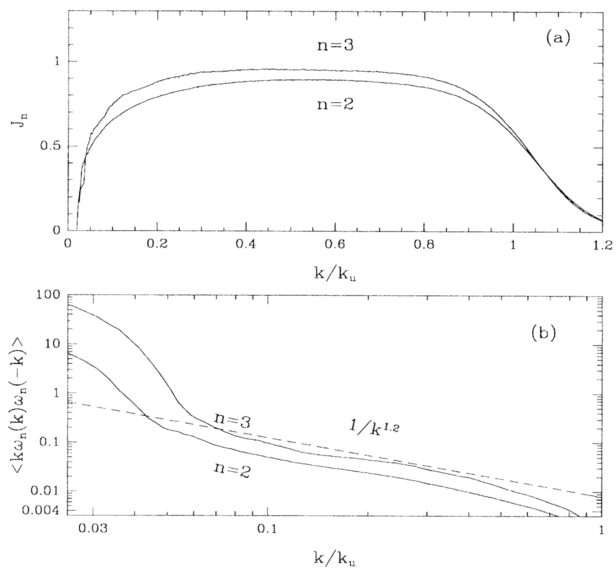

FIG. 5. Fluxes and spectra for higher powers of $\omega(x)^n$ for $n=2,3$ ($H2$ run): (a) fluxes J_n are normalized by $\eta\langle\omega^2\rangle^{n-1}(2n-1)!!$, for $n=2,3$; (b) log-log plot of spectra $\langle k\omega_n(k)\omega_n(-k)\rangle$, which are normalized by $\eta^{2/3}\langle\omega^2\rangle^{n-1}(2n-1)!!$, where $\omega_n(k)$ is the Fourier transform of $\omega(r)^n$, for $n=2,3$; dashed line $\sim 1/k^{1.2}$.

moments in physical space, such as vorticity differences $[\omega(r)-\omega(0)]^n$. First we checked the Kolmogorov-type relation for the velocity-vorticity third order correlator $[\Delta\omega(r)]^2\Delta v_n(r)=2\eta r_n$, which is a direct consequence of the Navier-Stokes equation. The results [see Fig. 4(d)] are consistent with the prediction up to $k_f r \lesssim 1$. Unfortunately, the vorticity differences mostly come from the region of spectrum very close to the force $k\approx k_f$, and give little useful information about the inertial range.

Apart from the enstrophy there are also higher conserved integrals of the type $I_n=\int\omega^{2n}(r)d^2r$. For these quantities it is possible to define corresponding fluxes J_n in complete analogy with the flux of vorticity (4). Whether we have constant or zero fluxes of I_n is unclear *a priori.* The measurements of J_n, along with the spectra of the Fourier transforms of the powers of vorticity $\omega(r)^n$ for $n=2,3$ are shown in Figs. 5(a) and 5(b), respectively. We indeed have the ranges of constant fluxes for these quantities. As may be expected for a Gaussian force, the absolute values of these fluxes scale as $\eta\langle\omega^2\rangle^{n-1}(2n-1)!!$ Within the range of constant flux, the spectra of ω^n are very close to $1/k$, as may be expected from "scalar field" type theories.

The main conclusion of this work is that the spectral exponent of the energy spectrum is quite close to -3. In contrast to the results here, other calculations for forced stationary homogeneous turbulence performed so far [9,10] yielded different anomalous dimensions. We think that one of the reasons for this discrepancy are the small Reynolds numbers in those simulations. We explicitly demonstrated how energy spectrum evolves toward $1/k^3$ when we increase Re_u, keeping Re_i fixed. We should emphasize that unlike [8] we have both direct enstrophy and inverse energy cascades. The analysis of our vorticity fields shows that we usually have 4–6 large coherent vortex structures for runs $H1$–$H3$ and around 10 vortices in the $H4$ case, where we have the largest infrared range ($k_f=14$). Therefore influence of vortices on direct enstrophy cascade decreases, while we increase direct cascade range. The data for the $H4$ run deviate from the Kraichnan law the most strongly, but the Reynolds number in this case is also the smallest. We may expect that the $H4$ run also will change it scaling toward $1/k^3$ after increasing Re_u. We cannot completely rule out the possibility that even for infinite Re_u the increase of the inverse cascade range may lead to anomalous dimensions in the enstrophy cascade range, but it seems unlikely at this point.

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