Vortex Dynamics and Superfluid Relaxation near the ^{4}He λ Transition

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The superfluid relaxation time near the ^{4}He λ transition is calculated using a vortex-ring theory of the phase transition. The diffusive response of the vortices is found by solving the Fokker-Planck equation, and the resulting relaxation time at low frequencies is in agreement with Landau-Khalatnikov theory and with dynamic scaling. At higher frequencies there is a deviation from the Landau-Khalatnikov form, with both the dissipation and superfluid density remaining finite at T_λ as the transition broadens at finite frequency.

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Considerable progress has been made in understanding the dynamics of phase transitions using the methods of dynamic scaling and perturbation expansions of model Hamiltonians [1]. When applied to the superfluid ^{4}He λ transition these techniques have provided in some cases a good description of experimental results [2]. However, the perturbation expansions are difficult to implement, and the physical interpretation of the results is often quite obscure. In this Letter a new approach to the dynamics of the superfluid λ transition is proposed, based on the dynamic response of the vortex rings that are the fundamental topological excitations underlying the phase transition [3–7]. The method is an extension of the two-dimensional calculation of Ambegaokar *et al.* [8] into three dimensions. The results for the superfluid relaxation time at low frequencies are found to be in full agreement with the Landau-Khalatnikov theory [9] and with dynamic scaling [1]. At higher frequency there is a deviation from the Landau-Khalatnikov form, with both the real and imaginary parts of the superfluid density remaining finite at T_λ as the transition broadens at finite frequency.

The response of a thermally excited vortex ring of average diameter a to a perturbing superflow $\mathbf{v}_s=\mathbf{v}_{so}\times\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$ is found by solving the Fokker-Planck equation [10] for the vortex distribution function Γ,

$$\frac{\partial\Gamma}{\partial t}=\frac{\partial}{\partial\mathbf{p}}\cdot\left[\Lambda k_BT\left(\frac{\partial\Gamma}{\partial\mathbf{p}}+\Gamma\frac{\partial F}{\partial\mathbf{p}}\right)\right], \tag{1}$$

where $p=\frac{1}{2}\pi^2(\hbar/m)\rho_s^0a^2$ is the momentum of the ring in a direction perpendicular to the plane of the ring, m is the helium mass, and ρ_s^0 is the unrenormalized superfluid density (taken to be equal [6] to the liquid density ρ). F is the energy (divided by k_BT) of the ring in the flow $\mathbf{v}_s$, $F=U(a)-g(k)\mathbf{p}\cdot\mathbf{v}_s/k_BT$, where $U(a)$ is the renormalized ring energy, and $g(k)$ is the finite-wave-number response function calculated in Ref. [6]. In the present paper we are interested in the long-wavelength limit $k\rightarrow 0$, and set $g(k)=1$. The equilibrium value of Γ with no flow is [8,10] $\Gamma_0=(a_0)^{-6}\exp(-U)$. The ring energy U is found from the static calculation of Refs. [3–7],

$$U(a)=\pi^2\int_{a_0}^{a}K_r[\ln(K^{-\theta})+1]\frac{da}{a_0}+\pi^2K_0C, \tag{2}$$

where $K_r=(\hbar^2/m^2)\rho_s a_0/k_BT$ with ρ_s the renormalized superfluid density, $K=K_r(a/a_0)$, K_0 is the initial value of K when the ring diameter equals the vortex core diameter a_0, and θ is the Flory exponent characterizing the self-avoiding random walk of the vortex loop, taken [4–7] to be $\theta=0.6$. The values of the core energy constant C and the core diameter were determined [6,7] to be $C=1.03$ and $a_0=2.3$ Å. Using the scaling relations of Ref. [6], K_r is found to fall to zero at the transition as a power law of the reduced temperature $t=(T_\lambda-T)/T_\lambda=1-K_{0c}/K_0$, and an exponent $\nu=0.67168\pm0.00003$ is found from fits [11] to the calculated values of K_r in the range $10^{-4}<t<10^{-6}$. This is in excellent agreement with Shenoy's prediction [4] $\nu=0.6717$.

The quantity Λ in Eq. (1) is the total drag force on the ring [10], proportional to its perimeter P. By computer simulations of the self-avoiding walk [12] the perimeter has been shown to scale with the average diameter a as

$$P/a_0=(a/a_0)^{1/(1-\theta)}. \tag{3}$$

A similar result was also found in Monte Carlo simulations [13] of the three-dimensional XY model. The drag force is then $\Lambda=\gamma P$, where γ is the mutual friction coefficient characterizing vortex-ring energy loss defined by Barenghi, Donnelly, and Vinen [14].

Assuming v_{so} is small, Eq. (1) can be linearized, and setting $\Gamma=\Gamma_0+\delta\Gamma e^{-i\omega t}$ gives an equation for $\delta\Gamma$,

$$-i\omega\delta\Gamma=\Lambda k_BT\frac{\partial^2\delta\Gamma}{\partial\mathbf{p}^2}+\frac{\partial\delta\Gamma}{\partial\mathbf{p}}\cdot\frac{\partial}{\partial\mathbf{p}}(\Lambda k_BT)+\frac{\partial}{\partial\mathbf{p}}\cdot\left[\Lambda\delta\Gamma\frac{\partial U}{\partial\mathbf{p}}\right]-\mathbf{v}_{so}\cdot\frac{\partial}{\partial\mathbf{p}}(\Lambda\Gamma_0). \tag{4}$$

Changing the variable from $\delta\Gamma$ to a response function $g(\omega,a)$ using

$$g(\omega,a)=\frac{3}{2}\frac{k_BT}{\Gamma_0pv_{so}}\int_0^\pi\delta\Gamma\cos\phi\sin\phi\,d\phi, \tag{5}$$

where ϕ is the angle between $\mathbf{p}$ and $\mathbf{v}_{so}$, Eq. (4) becomes

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$$2p^2\frac{\partial^2 g}{\partial p^2}+p\frac{\partial g}{\partial p}\left[\frac{21}{2}-\tilde{E}\right]+g\left[\frac{5}{2}-\tilde{E}+\frac{i\omega a_0^2}{D}\frac{\pi}{8}\left(\frac{a}{a_0}\right)^{\theta/(1-\theta)}\right]+\left[\frac{5}{2}-\tilde{E}\right]=0\,, \tag{6}$$

where $\tilde{E}=\pi^2 K_r(a/a_0)[\ln(K^{-\theta})+1]$ is related to the screened ring energy U, and like U it becomes a constant near T_λ ($\tilde{E}\sim 5$ for rings smaller than the coherence length). In Eq. (6) the diffusion constant D has been defined by $D=\gamma k_B T/\pi a_0(\rho_s^0 h/m)^2$, and as shown by Donnelly [15] this is the diffusion coefficient of the smallest ring of diameter a_0. Once g has been determined from Eq. (6), the frequency-dependent superfluid density is found from the susceptibility,

$$\mu(\omega)=\frac{\rho}{\rho_s(\omega)}=1+\int_{a_0}^{\infty}\frac{\partial\mu(0)}{\partial a}g(\omega,a)da\,. \tag{7}$$

In solving Eq. (6) a reasonable approximation is to neglect the derivatives of g (which are spatial derivatives since the ring momentum p varies as a^2). The equation is quite similar to that found in the two-dimensional case [8], where it was shown that neglecting the derivatives only leads to a redefinition of the diffusion coefficient by a constant factor of about 7. Making this same approximation with Eq. (6) gives

$$g(\omega,a)=\frac{\tilde{E}-5/2}{(-i\omega a_0^2/D')(\pi/8)(a/a_0)^{\theta/(1-\theta)}+(\tilde{E}-5/2)}\,, \tag{8}$$

where D' is assumed to be shifted from D by some constant (and will be determined below by comparison with experiment).

In the limit of low frequencies ($\omega\tau\ll 1$) the superfluid density computed from the response function of Eq. (8) is found to have the same form as predicted by the phenomenological Landau-Khalatnikov theory, $\rho_s(\omega)/\rho=[\rho_s(0)/\rho](1-i\omega\tau)$, where the relaxation time τ is given by

$$\tau=\frac{\pi}{8}\frac{a_0^2}{D'}\frac{\rho_s(0)}{\rho}\int_{a_0}^{\infty}\frac{\partial\mu(0)}{\partial a}\frac{(a/a_0)^{\theta/(1-\theta)}}{\tilde{E}-5/2}da\,. \tag{9}$$

An evaluation [16] of this expression using the static theory of Ref. [6] is shown in Fig. 1 as a function of the reduced temperature t. As predicted by Pokrovskii and Khalatnikov [9], the relaxation time diverges near T_λ as $t^{-1.0}$. A more careful fit of the calculated values by the form $\tau=\tau_0 t^{-x}$ for reduced temperatures between 10^{-4} and 10^{-6} yields $x=1.0075\pm 0.0001$, in precise agreement with the prediction of dynamic scaling [1], $x=3\nu/2$. The value of the exponent does not depend on the approximation made in going from Eq. (6) to Eq. (8) because in the limit of zero frequency g approaches 1 and the form of Eq. (8) becomes exact.

The value of τ_0 in Fig. 1 has been matched to the experimental result [17] $\tau_0=1.8\times 10^{-12}$ s, which requires $D'=1.2\times 10^{-5}$ cm^2/s. This is a reasonable order of magnitude for the vortex diffusion coefficient: Ambegaokar *et al.* [8] suggested on dimensional grounds that D should have a value of order $\hbar/m\approx 1\times 10^{-4}$ cm^2/s. A diffusion constant of the same order of magnitude has also been deduced from dynamic perturbation expansions [18] that uses thermal conductivity data as an input to determine the expansion parameters.

At higher frequencies ($\omega\tau>1$) the calculated $\rho_s(\omega)/\rho$ deviates from the Landau-Khalatnikov form. Figure 2 shows the real part of the superfluid density for several different frequencies. When $\omega\tau$ becomes greater than 1, $\rho_s(\omega)$ deviates from the static value, remaining finite at

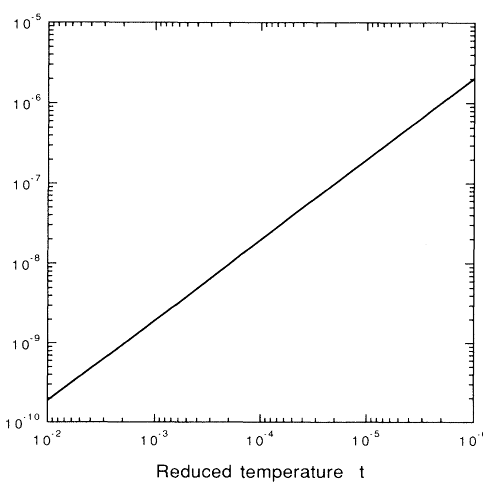

FIG. 1. The superfluid relaxation time τ calculated from Eq. (9) as a function of reduced temperature t.

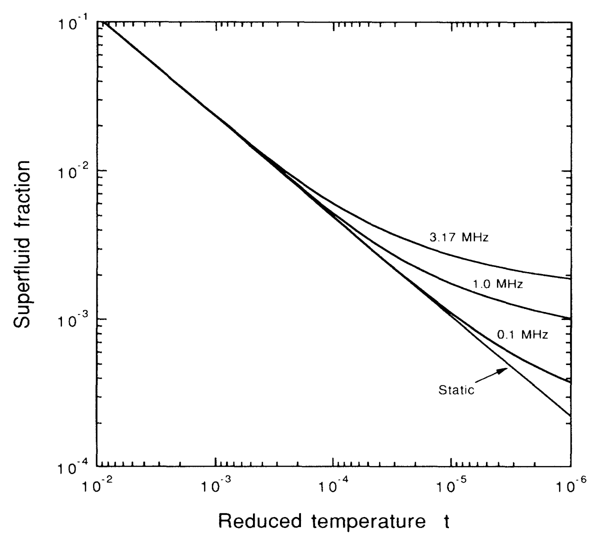

FIG. 2. The real part of the superfluid fraction $\rho_s(\omega)/\rho$ as a function of reduced temperature t, for several different frequencies ($\omega/2\pi$).

T_λ. This is a finite-frequency broadening of the transition, and just as in the two-dimensional case [8] it has a very simple interpretation as a finite-size broadening of the transition [5]. At finite frequencies the largest rings are overdamped and cannot respond to the applied superflow. But at zero frequency it is the response of the very largest rings that drives ρ_s exactly to zero at T_λ. Since g falls to zero in Eq. (7) at finite ring size, the largest rings do not contribute, and hence ρ_s remains finite at and above T_λ.

The broadening of the transition is also accompanied by dissipation, due to the rings that are maximally out of phase with the driving flow field. Figure 3 shows the magnitude of the imaginary part of the superfluid density, and this is proportional to the dissipation that would be observed, for example, in a torsion oscillator [8]. There is a peak in the dissipation at $\omega\tau = 1$, and then a decrease to a finite value at T_λ. This behavior is very similar to that observed in first sound attenuation [17], although the correspondence is not complete. First sound attenuation is related to the imaginary part of the time-dependent specific heat [18], rather than the imaginary part of ρ_s. Since the same response function of Eq. (8) will enter the time-dependent free energy [4], it is plausible that the calculated sound attenuation will be similar to Fig. 3, but this has yet to be verified.

The above results provide new insights into the nature of mutual friction, the drag force exerted on a vortex line moving with respect to the normal fluid. As first shown by Pitaevskii [19], the observed divergence [20] of the drag force near the λ point is related to the divergence of the superfluid relaxation time. In the present model the mutual friction results from an exchange of energy between the vortex line and the thermally excited vortex loops which constitute the normal fluid. As the line moves it perturbs the distribution of loops, and since they cannot adjust instantaneously, the interaction is dissipative. At low temperatures away from T_λ only the smallest loops of diameter a_0 are excited, and since they have a short relaxation time $\sim 10^{-12}$ s (Fig. 1), the mutual friction is relatively small. In this regime a connection with the semiclassical scattering theory of Hall and Vinen [21] can be made, since the smallest loops are identified as the roton excitations of the Landau theory [6,7]. As the temperature is raised larger loops are excited which cannot respond as rapidly, leading to the divergence of the mutual friction coefficients at the λ point.

In summary, the relaxation time of the superfluid density has been calculated using a vortex theory of the superfluid transition, and is found to be in agreement with known results. It should be possible to extend these results to a calculation of the superfluid response for arbitrary ω and k, since, to a good approximation, the complete response function $g(\omega,k)$ is simply the product of the response function found here and the finite-wavenumber response function of Ref. [6]. A further extension of this method to the propagation characteristics of first and second sound will require a calculation of the time-dependent specific heat. It should be remarked that the good agreement found here for the relaxation time provides additional evidence [12,22] supporting the Flory scaling proposal of Shenoy [4].

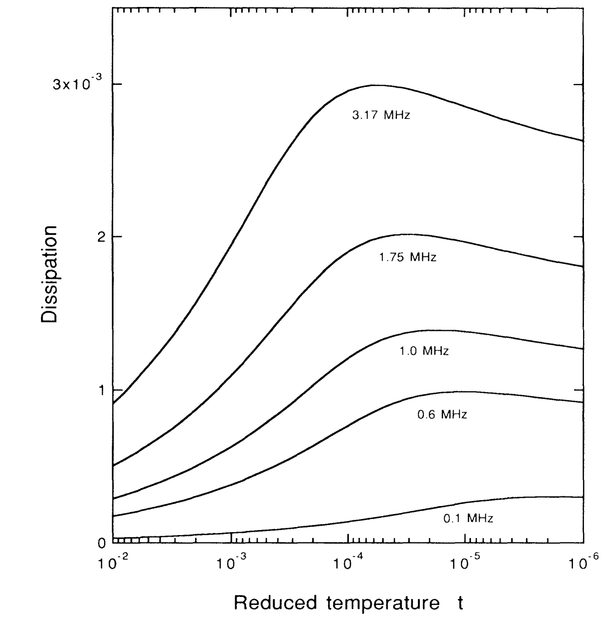

FIG. 3. The magnitude of the imaginary part of the superfluid fraction $\rho_s(\omega)/\rho$ (proportional to the dissipation) as a function of reduced temperature t, for different frequencies $(\omega/2\pi)$.

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