Instability of the Fixed Point of the $O(N)$ Nonlinear σ Model in $2+\epsilon$ Dimensions

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We calculate the full dimension (canonical plus anomalous), y_{2s}, of an infinite number of $O(N)$ invariant operators with $2s$ gradients. We find that for large s ($\epsilon = d-2$), $y_{2s} = 2(1-s) + \epsilon\{1 + [s^2/(N-2)][1+O(1/s)]\} + [\epsilon^2 s^3/(N-2)^2][\frac{2}{3} + O(1/s)] + O(\epsilon^3)$, correct to two-loop order. Thus, in two-loop order y_{2s} grows even more rapidly than in one-loop order. Even if ϵ is arbitrarily small, one can always find, to two-loop order, operators with positive full dimension by choosing s sufficiently large. We argue that the conventional analysis of this problem may be inadequate.

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While exploring the quantum effects in disordered electronic systems, Kravstov, Lerner, and Yudson [1] discovered a remarkable phenomenon. In the formalism of $2+\epsilon$ expansion, there exists an infinite number of relevant high gradient operators (HGOs) connected to the nonlinear σ model describing the Anderson localization problem of an electron in a random potential. Although such operators are irrelevant by power counting, the *one-loop* term renders them relevant, more so as the number of gradients increases. This nonlinear σ model defined on a Grassmannian manifold arises naturally within the framework of the replica field theory. The intimate connection between the anomalous dimension of the HGOs and the asymptotics of the distribution of conductance fluctuations has been extensively discussed in the literature [2]. On the other hand, in a *one-loop* calculation, it has been recently shown by Wegner [3] that a similar situation exists for the $O(N)$ nonlinear σ model which is believed to describe the critical properties of the classical (in the sense of statistical mechanics) Heisenberg model [4]. It is also worth noting that there exists a close analogy between the conductance fluctuations and the fluctuations of the spin stiffness constant, as was shown recently by one of us [5].

The existence of an infinite number of relevant operators at the nontrivial fixed point is, to say the least, worrisome. Thus, in this paper we examine the $O(N)$ model and calculate the full dimension of an infinite number of $O(N)$ invariant operators with $2s$ gradients to *two-loop* order. We discuss the method of our calculation, the final results, and the physical implications. We conclude that the conventional analysis of this problem may be inadequate. The actual computations are too long to be described here and will be presented elsewhere in detail [6]. The present calculation would have been virtually impossible without the techniques discussed by Alvarez-Gaumé, Freedman, and Mukhi [7] and Grisaru, Van de Ven, and Zanon [8]. Thus, the excursion to the Riemannian manifold below is a necessity and not a luxury. If the two-loop term is nonvanishing, a similar calculation should be feasible for the localization problem. Unfortunately, for the localization problem there is some suspicion [9] that both the two- and the three-loop terms are vanishing; that is, the next contribution comes from the four-loop term. This is a suspicion, however. The reader should not prematurely despair.

The action of the $O(N)$ nonlinear σ model can be written as

$$S = \frac{1}{2}\int d^d x\, g_{mn}(\pi)\partial_\alpha \pi^m \partial^\alpha \pi^n , \tag{1}$$

where the metric $g_{mn}(\pi)$ is given by $t g_{mn}(\pi) = \delta_{mn} + \pi_m \pi_n/(1-\pi^2)$. The coupling constant t (the temperature) is absorbed in the definition of the metric. The space $O(N)/O(N-1)$ is a Riemannian manifold with the coordinates $\pi^n(x)$ and the metric $g_{mn}(\pi)$. In this form, the action is invariant under general coordinate transformations (parametrizations) on the manifold. This formulation leads to considerable simplifications in the calculations described below. Conventionally, the action is written in terms of an N-component spin vector $\mathbf{S} \equiv (\sigma, \pi^1, \ldots, \pi^{(N-1)})$. The action above is obtained by resolving the constraint $|\mathbf{S}| = 1$.

We calculate the anomalous dimensions of the high gradient operators, $O_{\rm HGO}$, defined by the cyclic products, such as

$$O_{\rm HGO} = (g_{m_1 n_1}\partial_{\mu_1}\pi^{m_1}\partial_{\mu_2}\pi^{n_1}) \cdots (g_{m_r n_r}\partial_{\mu_r}\pi^{m_r}\partial_{\mu_1}\pi^{n_r}) , \tag{2}$$

where the total number of gradients is $2s$. We determine the renormalization of these operators, O_j, by calculating the matrix $Z_{ij}(t,\epsilon)$ in the following equation:

$$O_i^R = \sum_j Z_{ij} O_j , \tag{3}$$

where O_i^R is the renormalized operator. Given the matrix Z_{ij}, we calculate

$$\gamma_{ij}(t) = \left(Z\mu \frac{d}{d\mu} Z^{-1} \right)_{ij} , \tag{4}$$

where the momentum μ is the renormalization scale. The largest eigenvalue of $\gamma_{ij}(t)$ evaluated at the fixed point is the anomalous dimension. We use the background field method and the dimensional regularization scheme. Thus, no redundant operators are generated during the renormalization process [3].

In the background field method, the field π^m is written

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as $\pi^m = \psi^m + \eta^m(\xi)$, where ψ^m are chosen to satisfy the classical equation of motion; η^m represent the fluctuations. Furthermore, to take full advantage of the symmetries, we express the fluctuations in terms of the normal coordinates ξ^m, i.e., $\eta^m(\xi)$ [7,8]. These coordinates are defined by the tangent vector at ψ^m of the geodesic that connects the points ψ^m and $\psi^m + \eta^m$. Then, the normal coordinates transform as vectors under general coordinate transformations. Consequently, the coefficients of the power series expansion of the HGOs and the action, in terms of ξ^m, are tensors. Therefore, by expanding the action and the HGOs around the classical fields ψ^m, and integrating over the new fields ξ^m, one generates a covariant diagrammatic expansion [7,8] that gives directly the invariant divergences. As a result the expectation value of an operator $\langle O \rangle$ can be expressed as

$$\langle O \rangle = O^{(0)}(\psi) + \frac{\int [d\xi] O^{(e)}(\psi,\xi) e^{-S^{(e)}(\psi,\xi)}}{\int [d\xi] e^{-S^{(e)}(\psi,\xi)}} , \tag{5}$$

where the terms $O^{(e)}(\psi,\xi)$ and $S^{(e)}(\psi,\xi)$ denote the second and the higher order terms of the power series in ξ^m. Note that the second term on the right-hand side will depend on ψ after the integration.

A simple rescaling of the field $\xi^m(x)$ shows that to compute the divergences up to order $O(t^2)$ we need to know the expansion of the action and the HGOs up to order $O(\xi^4)$. If we define $G_{\mu\nu}(\pi)$ to be $g_{mn}\partial_\mu \pi^m \partial_\nu \pi^n$, the first four orders of the expansion of $G_{\mu\nu}(\pi)$ are given by

$$G^{(2)}_{\mu\nu}(\pi) = g_{mn} D_\mu \xi^m D_\nu \xi^n + R_{mk_1k_2n} \partial_\mu \psi^m \partial_\nu \psi^n \xi^{k_1} \xi^{k_2} , \tag{6}$$

$$G^{(3)}_{\mu\nu}(\pi) = \tfrac{2}{3} R_{mijk} \partial_\mu \pi^m D_\nu \xi^i \xi^j \xi^i + (\mu \leftrightarrow \nu) , \tag{7}$$

$$G^{(4)}_{\mu\nu}(\pi) = \tfrac{1}{3} R_{pijm} R^p_{kln} \partial_\mu \psi^m \partial_\nu \psi^n \xi^i \xi^j \xi^k \xi^l + \tfrac{1}{3} R_{nijm} \xi^i \xi^j D_\mu \xi^m D_\nu \xi^n , \tag{8}$$

where the covariant derivative $D_\mu \xi^m$ is $D_\mu \xi^m \equiv \partial_\mu \xi^m + \Gamma^m_{st} \partial_\mu \psi^t \xi^s$; Γ is the Christoffel symbol and R is the well-known curvature tensor. The corresponding expansion of the action can be easily obtained from Eqs. (6)–(8). The expression for $G^{(2)}_{\mu\mu}$ shows that the quadratic part of the action has the nonstandard form $g_{mn}(\psi) \times \partial_\mu \xi^m \partial^\mu \xi^n$. The well-known quadratic form $\partial_\mu \xi_i \partial^\mu \xi^i$ is obtained with the transformation of coordinates $\xi^a = e^a_j(\psi)\xi^j$, where $e^a_m(\psi)$ is the vielbein [10]. The substitution of this new transformation yields

$$e^a_m D_\mu \xi^m \equiv D_\mu \xi^a = \partial_\mu \xi^a + \omega^a_{mb}(\psi) \partial_\mu \psi^m \xi^b ,$$

where ω^{ab}_m is the spin connection. The only property of ω^{ab}_m that is necessary in this paper is the one satisfied by the quantity $A^{ab}_\mu(\psi) \equiv \omega^{ab}_m \partial_\mu \psi^m$. It can be shown that A^{ab}_μ transforms like a gauge field under the rotations that connect the previously defined local coordinate systems [10]. So, it is not surprising that A^{ab}_μ satisfies the following identity:

$$\partial_\mu A^{ab}_\nu - \partial_\nu A^{ab}_\mu + A^{ac}_\mu A^{cb}_\nu - A^{ac}_\nu A^{cb}_\mu = e^{ai} e^{bj} R_{ijmn} \partial_\mu \psi^m \partial_\nu \psi^n \equiv R^{ab}_{mn} .$$

This identity is important, because it is the only covariant quantity that can be formed from the gauge field $A^{ab}_\mu(\psi)$. Any divergence which has the gauge field $A^{ab}_\mu(\psi)$ must have this form. To find the counterterms that form $Z_{ij}(t,\epsilon)$, we follow the Bogoliubov-Parasiuk-Hepp-Zimmermann procedure [11] and subtract the subdivergences of the divergent graphs, and then extract the poles resulting from the subtraction. To regulate the infrared divergences and to determine correctly the ultraviolet divergences, we introduce a mass cutoff m^2 in each propagator, i.e., we write $p^2 + m^2$ instead of p^2.

The calculation of the eigenvalues of γ_{ij} is easier if we introduce the conformal coordinates, $z = x + iy$, $\bar{z} = x - iy$, because the classification of the HGOs is simpler [1]. In conformal coordinates any HGO is given by

$$O_{\mathrm{HGO}} = (g_{mn} \partial_z \pi^m \partial_{\bar{z}} \pi^n)^{s-2p} (g_{ij} \partial_{\bar{z}} \pi^i \partial_{\bar{z}} \pi^j)^p \times (g_{kl} \partial_z \pi^k \partial_z \pi^l)^p , \tag{9}$$

where $p = 0, 1, \ldots, [s/2]$, and $[a]$ is the integer part of a. Detailed two-loop computations lead to

$$\gamma(t) = -At - Bt^2 + O(t^3) , \tag{10}$$

where the matrices A and B are

$$A = \begin{bmatrix} (C^{(1)}) & (0) \\ (0) & (D^{(1)}) \end{bmatrix}, \quad B = \begin{bmatrix} (C^{(2)}) & (X) \\ (Y) & (D^{(2)}) \end{bmatrix} . \tag{11}$$

The basis space for the blocks $C^{(1)}$ and $C^{(2)}$ are the high gradient operators. In the one-loop matrix A, this set of operators is closed under the renormalization group transformations. This, however, is not true for the two-loop matrix B, but, as discussed below, it is sufficient to calculate the matrices $C^{(1)}$ and $C^{(2)}$. The matrices $C^{(1)}$ and $C^{(2)}$ are

$$C^{(1)} = \frac{1}{2\pi} \begin{bmatrix} q_{11} & q_{12} & 0 & 0 & \cdots \\ 0 & q_{22} & q_{23} & 0 & \cdots \\ 0 & 0 & q_{33} & q_{34} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad C^{(2)} = \frac{1}{12\pi^2} \begin{bmatrix} r_{11} & r_{12} & 0 & 0 & 0 & \cdots \\ r_{21} & r_{22} & r_{23} & 0 & 0 & \cdots \\ 0 & r_{32} & r_{33} & r_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} , \tag{12}$$

where

$$q_{j+1\,j+1} = (s-2j)(s-2j-1) - 2\nu j , \tag{13}$$

$$q_{j+1\,j+2} = -(s-2j)(s-2j-1) , \tag{14}$$

$$r_{j+1\,j+1}=2(s-2j)(s-2j-1)(s-2j-2)+\tfrac{1}{2}(s-2j)(s-2j-1)(v+6)+3v(s-2j)$$
$$-2j(s-2j)(s-2j-1)+14j^2(s-2j)+6vj(s-2j)-6vj-3vj(j-1)-j^2(v-9)+4j^2(j-1)\,, \quad (15)$$

$$r_{j+1\,j+2}=-2(s-2j)(s-2j-1)(s-2j-2)-\tfrac{1}{2}(s-2j)(s-2j-1)(6-13v/2)+2j(s-2j)(s-2j-1)\,, \quad (16)$$

$$r_{j+1\,j}=-14j^2(s-2j)-4j^2(j-1)+j^2(v-9)\,, \quad (17)$$

$j=0,1,2,\ldots,[s/2]$, and $v=(N-2)$.

The largest eigenvalue of the matrix A lies in the block $C^{(1)}$ and was obtained by Wegner [3]. This gives the one-loop result for the anomalous dimension. To calculate the anomalous dimension correct to two-loop order we use elementary first order nondegenerate perturbation theory (the eigenvalues are all distinct). The first order correction to the eigenvalues are contained in the diagonal elements of the matrix

$$\begin{pmatrix} \alpha C^{(2)}\alpha^{-1} & \alpha X\beta^{-1} \\ \beta Y\alpha^{-1} & \beta D^{(2)}\beta^{-1} \end{pmatrix},$$

where the matrix $\begin{pmatrix} \alpha & 0 \\ \delta & \beta \end{pmatrix}$ diagonalizes the first order matrix A. We calculate only the block $\alpha C^{(2)}\alpha^{-1}$; it is not necessary for us to know the block $\beta D^{(2)}\beta^{-1}$. Because the largest full dimension belonging to the upper block is found to be positive, we already have at our disposal an infinite set of operators with positive full dimension. Even if the eigenvalues in the lower block are larger (an unlikely event, as can be seen from perturbation theory), our conclusions are unchanged. This explains why, throughout our calculation, we have not been concerned about the mixing of the HGOs with other classes of operators in the two-loop approximation. As mentioned before, such mixings do take place at the two-loop level. Thus the anomalous dimensions of the HGOs at the fixed point $t=t^*=[2\pi\epsilon/(N-2)][1-\epsilon/(N-2)]$, in the limit of large s, are

$$y_{2s}=2(1-s)+\epsilon\{1+[s^2/(N-2)][1+O(1/s)]\}$$
$$+[\epsilon^2 s^3/(N-2)^2][\tfrac{2}{3}+O(1/s)]+O(\epsilon^3)\,. \quad (18)$$

If the positivity of y_{2s} for large s continues to hold to higher loop orders, we have an unusual situation. The fixed point is infinitely unstable. Moreover, it can be shown [3] that there is no feedback from the operators with a higher number of gradients to those with a lower number of gradients to all orders in perturbation theory. The situation is entirely different from the ϵ expansion of the ϕ^4 theory around four dimensions. Unlike ϕ^4 theory, the present formalism does not allow us to determine the new fixed point because of the lack of feedback. Last, we note that in Eq. (18) the limits $N\to\infty$ and $s\to\infty$ do not commute.

It is also interesting to write down the renormalization group β functions. Defining e^l to be the length rescaling factor and U_{2m} to be the charges associated with the HGOs,

$$\frac{dt}{dl}=-(d-2)t+(N-2)\frac{t^2}{2\pi}+(N-2)\frac{t^3}{(2\pi)^2}+\cdots, \quad (19)$$

$$\frac{1}{U_{2s}}\frac{dU_{2s}}{dl}=d-2s+\frac{ts(s-1)}{2\pi}+\frac{t^2[s^3+O(s^2)]}{12\pi^2}+\cdots. \quad (20)$$

The renormalization group flows for $d=2$ and $d=2+\epsilon$ are shown in Figs. 1 and 2.

Strictly speaking, the strong coupling behavior cannot be obtained from these renormalization group equations. We examine $d=2+\epsilon$ first. There are two principal possibilities for $t>t^*$: (1) $U_{2s}\to 0$ and (2) $U_{2s}\to\infty$. The first implies that the magnetization drops discontinuously to zero at the phase transition, but that the correlation length tends to ∞ [12]. The second possibility would mean that, within the framework of ϵ expansion, the system cannot disorder. Although the increase of t signifies disorder, simultaneous increase of an infinite number of U_{2s} would not allow the spins to deviate from a fixed direction. This possibility is reminiscent of an idea due to Halperin [13] who argued that topological defects may be necessary to disorder the system for $N\leq d$. Note, however, that the present formalism should hold for arbitrary N and d. Nonetheless, we feel that the breakdown of the perturbative renormalization group would be similar in spirit to that discussed by Halperin.

We are in similar trouble for $d=2$. The simplest possi-

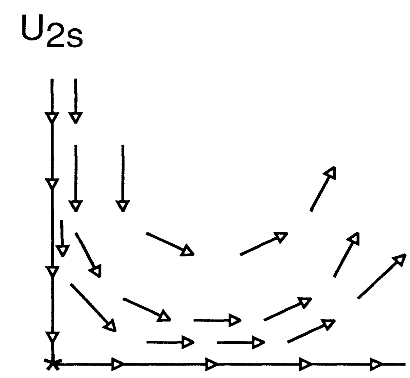

FIG. 1. The renormalization group flows for $d=2$. U_{2s} is the charge associated with one of the high gradient operators for sufficiently large s.

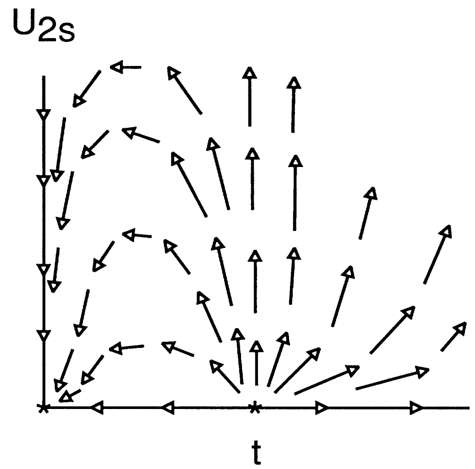

FIG. 2. The renormalization group flows for $d=2+\epsilon$. U_{2s} is the charge associated with one of the high gradient operators for sufficiently large s.

ble escape from our dilemma is as follows. Note that at low temperatures U_{2s}'s tend to get smaller until the temperature is sufficiently large that they start growing again, which depends, of course, on the precise value of s. Thus, for low enough temperatures, the renormalization group equations are at least *approximately* valid. Now, if it is possible, within the range of their approximate validity, to match the solutions of the low temperature renormalization group equations to strong coupling calculations (for example, a Monte Carlo or a series expansion), then the perturbative renormalization group analysis can still serve a useful purpose [14]. Even in $d=2$ the growth of U_{2s} at higher temperatures can be interpreted to mean that the system cannot disorder within the perturbative scheme and that the instantons are necessary [15]. The other possibility of $U_{2s}\rightarrow 0$ at even higher temperatures appears to be unlikely but cannot be ruled out.

In conclusion, we note that the extrapolation of the $2+\epsilon$ expansion to $\epsilon=1$ does not seem to reproduce what we believe to be the correct critical behavior of the lattice Heisenberg model at $d=3$. Nor does this extrapolation seem to match smoothly that obtained from four dimensions using ϕ^4 theory [16]. Presently the *three-loop* calculation appears to be barely feasible. As it stands, uncritical use of $2+\epsilon$ expansion may be misleading, more so in the Anderson localization problem where there are not many reliable checks. The results for $d=2$ can be salvaged only if they are supplemented by strong coupling calculations [14].

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