Controlling Chaos in Systems Described by Partial Differential Equations

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(Received 11 June 1993)

A model one-dimensional drift-wave equation driven by a sinusoidal wave is used to study controlling chaos in partial differential equations. By injecting negative feedback through a monochromatic wave or pinning at a certain x-space point we can successfully stabilize unstable steady states and control chaos. In the case of bistability, an unstable steady state in the middle branch can be also stabilized by these controlling approaches. The methods used in this Letter can be applied to general continuous spatiotemporal systems.

PACS numbers: 05.45.+b, 52.35.Kt

Recently there have been considerable efforts to control chaos [1-8]. Among various approaches feedback control turns out to be particularly interesting [1-4,7,8]. However, most of the works have focused on temporal systems. Since the dynamics of spatiotemporal systems is more complicated and richer, it is an important step to extend the study of controlling chaos to systems described by partial differential equations, which are extensively used in the study of chaos in hydrodynamics, plasma physics, laser physics, and so on [9,10]. The present Letter attacks this goal by numerical simulations.

We use a one-dimensional nonlinear drift-wave equation driven by a sinusoidal wave as our model [11,12],

$$\frac{\partial\phi}{\partial t}+a\frac{\partial^3\phi}{\partial t\partial x^2}+c\frac{\partial\phi}{\partial x}+f\phi\frac{\partial\phi}{\partial x}=-\gamma\phi-\epsilon\sin(x-\Omega t)\,. \tag{1}$$

The system is, in its own right, important in plasma physics and nonlinear science [11-14]. In Eq. (1), a 2π-periodic condition $[\phi(x+2\pi,t)=\phi(x,t)]$ is used. Setting $z=x-\Omega t$, we can rewrite Eq. (1) in the frame of the driving wave as

$$\frac{\partial}{\partial t}\left(1+a\frac{\partial^2}{\partial z^2}\right)\phi=\Omega\frac{\partial}{\partial z}\left(1+a\frac{\partial^2}{\partial z^2}\right)\phi-c\frac{\partial\phi}{\partial z}-f\phi\frac{\partial\phi}{\partial z}-\gamma\phi-\epsilon\sin z\,. \tag{2}$$

Throughout the Letter we fix the dispersion and dissipation as $a=-0.287$, $\gamma=1.0$, and the coefficients of the gradient and nonlinear terms as $c=1.0$, $f=-6.0$. The strength and frequency of the driving wave, ϵ and Ω, are the only changeable parameters. An integral $E(t)$, which is the well known energy of the system,

$$E(t)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}[\phi^2(x,t)-a\phi_x^2(x,t)]dx\,, \tag{3}$$

is conveniently used to monitor the dynamics of the system. By setting $\partial/\partial t=0$ we can get a steady state solution of Eq. (2), $\phi=\hat{\phi}(z)$, which is a running wave in x space, $\hat{\phi}(z)=\hat{\phi}(x-\Omega t)$.

It is convenient to divide the solution of Eq. (2) into two parts, $\phi(z,t)=\hat{\phi}(z)+\delta\phi(z,t)$ and to expand $\delta\phi(z,t)$ as $\delta\phi(z,t)=\lim_{N\to\infty}\sum_{k=1}^{N}b_k(t)\cos(kz+\alpha_k(t))$. Equation (2) for $\delta\phi$ can be transformed to a set of ordinary differential equations

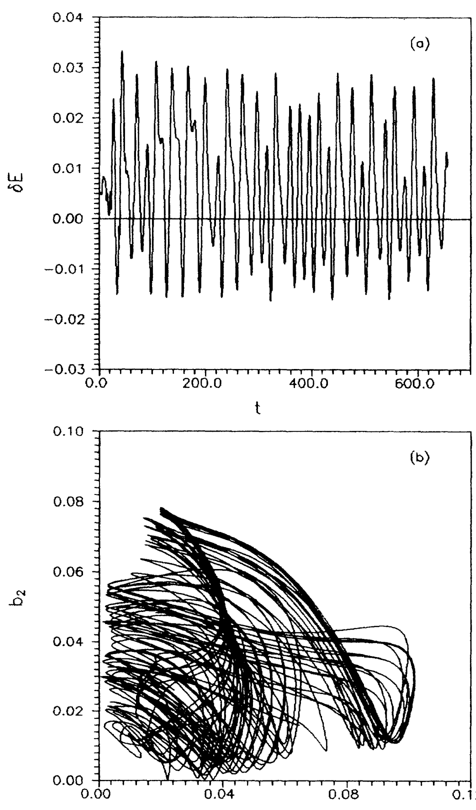

FIG. 1. Asymptotic motion of Eqs. (4) is chaotic for $\Omega=0.65$, $\epsilon=0.2$. (a) $\delta E(t)$ vs t. (b) Trajectory in the b_1-b_2 plane.

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$$\dot{b}_k(t) = N_k(t), \quad \dot{\alpha}_k(t) = M_k(t)\,; \tag{4}$$

here

$$\begin{aligned}
N_k(t) = &-\frac{\gamma}{1-ak^2}b_k + \frac{kf}{2(1-ak^2)}\\
&\times\Bigg(\sum_{l+l'=k}[A_l b_{l'}\sin(\theta_l+\alpha_{l'}-\alpha_k)+b_l b_{l'}\sin(\alpha_l+\alpha_{l'}-\alpha_k)]\\
&+\sum_{l-l'=k}[A_l b_{l'}\sin(\theta_l-\alpha_{l'}-\alpha_k)+b_l b_{l'}\sin(\alpha_l-\alpha_{l'}-\alpha_k)]\\
&+\sum_{l'-l=k}[A_l b_{l'}\sin(-\theta_l+\alpha_{l'}-\alpha_k)+b_l b_{l'}\sin(-\alpha_l+\alpha_{l'}-\alpha_k)]\Bigg)\,,\\
M_k(t) = &-k\left(\frac{c}{1-ak^2}-\Omega\right)-\frac{kf}{2(1-ak^2)b_k}\\
&\times\Bigg(\sum_{l+l'=k}[A_l b_{l'}\cos(\theta_l+\alpha_{l'}-\alpha_k)+b_l b_{l'}\cos(\alpha_l+\alpha_{l'}-\alpha_k)]\\
&+\sum_{l-l'=k}[A_l b_{l'}\cos(\theta_l-\alpha_{l'}-\alpha_k)+b_l b_{l'}\cos(\alpha_l-\alpha_{l'}-\alpha_k)]\\
&+\sum_{l'-l=k}[A_l b_{l'}\cos(-\theta_l+\alpha_{l'}-\alpha_k)+b_l b_{l'}\cos(-\alpha_l+\alpha_{l'}-\alpha_k)]\Bigg)\quad (k=1,2,\ldots,N)\,,
\end{aligned}$$

where A_k and θ_k are given by the steady solution $\hat{\phi}(z) = \lim_{N\to\infty}\sum_{k=1}^{N} A_k\cos(kz+\theta_k)$. In numerical simulations N is chosen such that the modes $k>N$ have negligible influence on the dynamical behaviors. For instance, at $\Omega=0.65$, $\epsilon=0.2$, Eq. (2) has three steady state solutions. We focus on the steady state with the lowest energy $\hat{E}$. An algebraic computation shows

$$A_1 = 0.11, \ldots, A_8 = 1.8\times10^{-3}, \ldots, A_{13} = 1.8\times10^{-5}.$$

Therefore, high modes play a negligible role. For the parameters chosen, $N=13$ is a very good approximation, therefore a set of 26 ordinary differential equations is dealt with in this Letter. N's up to 20 have been tried and give no visible change either to the steady solutions or to the time-dependent behavior of the system. At the given Ω,ϵ values the above mentioned steady state is unstable, and the motion of the system is chaotic. In Fig. 1 we run Eqs. (4) by taking the initial state in the vicinity of the given unstable steady state. Figure 1(a) shows $\delta E = E(t) - \hat{E}$ vs t. The energy changes with time violently. In Fig. 1(b) we plot the projection of the stationary trajectory of Eqs. (4) in the b_1-b_2 plane. The motion is obviously chaotic.

A direct way to stabilize $\hat{\phi}(z)$ is to force Eq. (2) by a negative feedback $-\lambda[\phi(z,t)-\hat{\phi}(z)]$. However, this feedback is difficult to realize practically since one has to monitor each point in x space (or each wave in k space) and respond to the system changes everywhere (or for every wave number). There are two convenient ways to control the system: The first is to monitor a single mode (e.g., $k=i$) and then to input a monochromatic wave $-\lambda b_i(t)$ to control the system; the second is to monitor a local region in x space and to apply feedback to the system in this region (a practical way may be to add a term like $-(\lambda/\pi r)[\phi(x,t)-\hat{\phi}(x-\Omega t)]\exp[-(x-x_0)^2/r]$, $0<r\ll1$; for numerical simplicity we can take the limit $r\to0$ and use $-\lambda\delta(x-x_0)[\phi(x,t)-\hat{\phi}(x-\Omega t)]$; the essence of the control is not changed by this limit). In the former case the ith equation in Eqs. (4) is changed to

$$\dot{b}_i(t) = N_i(t) - \lambda b_i(t)/(1-ai^2)\,, \tag{5}$$

and all the other equations for $b_{k\neq i}(t)$ and $\alpha_k(t)$ remain the same as in Eqs. (4). In the latter case Eqs. (4) are replaced by

$$\begin{aligned}
&\dot{b}_k(t) = N_k(t) - \lambda g_k(t), \quad \dot{\alpha}_k(t) = M_k(t) - \lambda h_k(t)\,,\\
&g_k(t) = \frac{1}{1-ak^2}\sum_{l=1}^{N} b_l(t)\cos[(l-k)(x_0-\Omega t)+\alpha_l-\alpha_k]\,,\\
&h_k(t) = \frac{1}{1-ak^2}\sum_{l=1}^{N} b_l(t)\sin[(l-k)(x_0-\Omega t)+\alpha_l-\alpha_k]/b_k, \quad k=1,2,\ldots,N\,.
\end{aligned} \tag{6}$$

Of the feedbacks Eqs. (5) and (6), one is a local input in the wave number space, the other is a local input in x space. To perform a local control in x space, one can insert an electrode at a given point x_0. However, for k-space control one has to make a spatial Fourier transformtion of the field, and work out variations of the desired harmonic wave. Thus, monitoring in k space must be more difficult than in x space. Nevertheless, as for response to the system, it would be

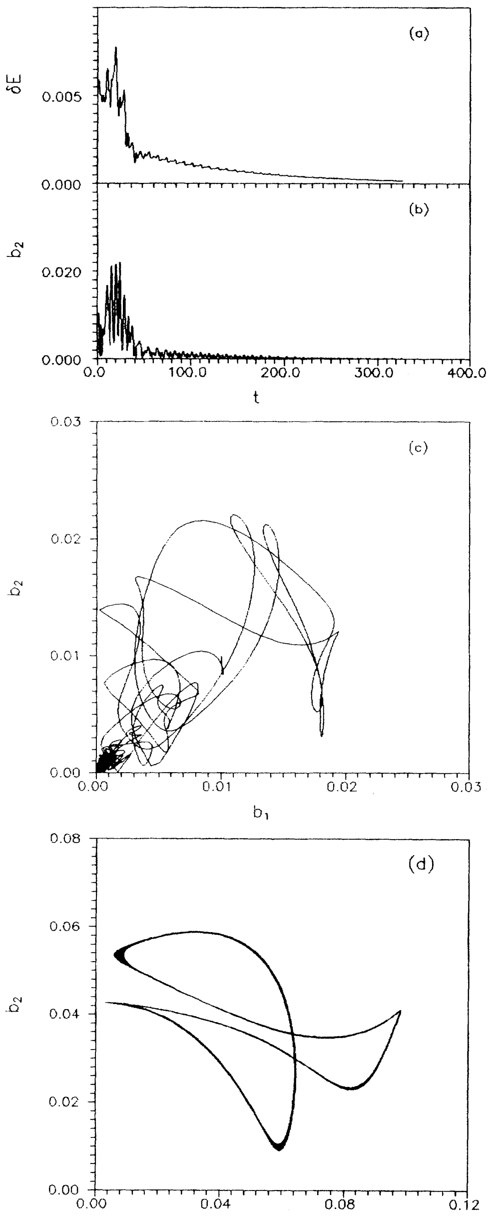

FIG. 2. Feedback is applied as in Eq. (5) with $i=2$, $\lambda=1.0$ for (a)–(c), $i=3$, $\lambda=1.0$ for (d); $\Omega=0.65$, $\epsilon=0.2$. (a) δE vs t. (b) $b_2(t)$ vs t. The forcing approaches zero as $t\to\infty$. (c) Trajectory in the b_1-b_2 plane. (d) Trajectory in the b_1-b_2 plane when $i=3$ is feedbacked. In this case $\hat{\phi}(z)$ is still unstable while chaos is replaced by a periodic motion.

sufficient in certain cases to inject one or few harmonic waves, as can be seen in the following, that may be easily performed.

Let us first control the system with $\Omega=0.65$, $\epsilon=0.20$ by injecting a monochromatic wave [$i=2$, $\lambda=1.0$ in Eq. (5)]. In Figs. 2(a) and 2(b) we plot $\delta E, b_2(t)$ against t, respectively. In Fig. 2(c) we show the trajectory in the

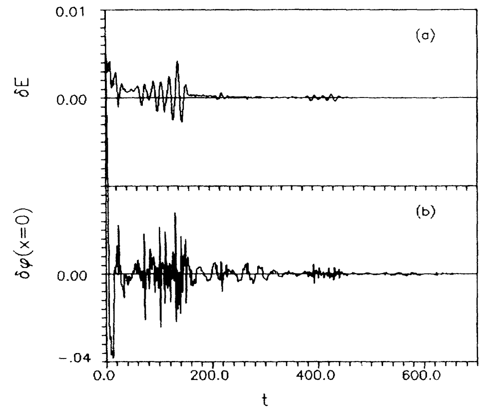

FIG. 3. The system is controlled by using Eqs. (6) with $x_0=0$, $\lambda=0.2$; $\Omega=0.65$, $\epsilon=0.2$. (a) δE vs t. (b) $\phi(0,t) - \hat{\phi}(x-\Omega t)|_{x=0}$ vs t.

b_1-b_2 plane. The initial state is far from the steady state. By injecting the monochromatic wave the system eventually approaches the aim state and the chaos is completely controlled. The perturbation is reduced to zero as time increases. It is remarkable that the unstable state of such a 26-dimensional system can be stabilized by forcing a single equation, and the chaotic motion of the partial differential equation can be controlled by simply inputting a single wave. We have tested many different parameters. There are various typical responses of the system to the external forcing acting on different wave numbers. First, there may be only a lower threshold of λ, say $\lambda=\lambda_L$, such that when $\lambda>\lambda_L$ the steady state is always stabilized. This is the case for $\Omega=0.65$, $\epsilon=0.20$, if we simultaneously input two modes, $-\lambda b_i(t)/(1-ai^2)$, $i=1,2$, to the first and second equations of Eqs. (4), respectively. Second, there are both limits λ_L and λ_U, such that the atom state can be approached only if $\lambda_U>\lambda>\lambda_L$. This is the case at the same Ω,ϵ as we input the monochromatic wave of either $i=1$ or $i=2$. For $i=1$ we find $\lambda_L\approx 0.5$, $\lambda_U\approx 3.0$. Third, the reference state can never be stabilized whatever λ is. This is the case if we feedback the system in Fig. 1 with a single mode $i\geq 3$. We expect that injecting multiple waves might be necessary for controlling chaos if more unstable modes appear at the reference state. Nevertheless, the number of injecting waves can be much less than the system dimension. The response of the system to the feedback is very interesting even if the forcing fails to stabilize the reference state. At different parameter combinations and forcings we have observed very rich patterns such as steady states (not the aim state), periodic motions, quasiperiodic motions, and so on. In most of the cases the chaos level of the system is effectively reduced though the reference state remains unstable. For example, in Fig. 2(d) we take the same Ω, ϵ, and λ as in 2(c) while feedback to the system is via the mode $i=3$, and plot the asymptotic trajectory also in b_1-b_2 space. A perfect

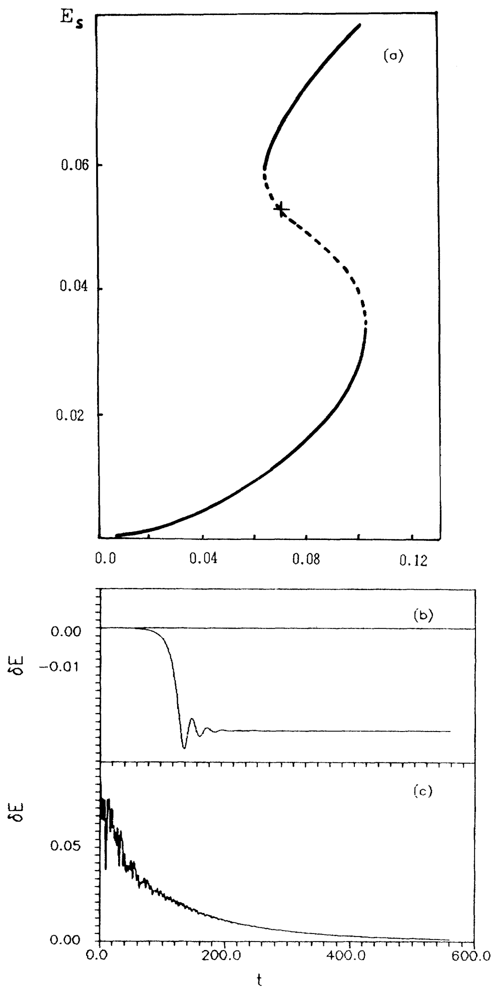

FIG. 4. Controlling of the middle branch. (a) E_s vs ϵ for $\Omega=0.56$; the middle branch is unstable. (b) $\delta E=E(t)-\hat{E}$ vs t without control, $\Omega=0.56$, $\epsilon=0.07$ [the cross in (a)]; here $\hat{E}$ is the energy of the middle branch. (c) δE vs t with control of Eqs. (5) ($i=2$, $\lambda=1.0$), $\Omega=0.56$, $\epsilon=0.07$. The system is locked to the middle branch eventually.

periodic motion is introduced by the forcing.

Now let us control chaos by using Eqs. (6), i.e., by pinning the system to the reference state at $x=x_0$. We take $x_0=0$, $\lambda=0.2$, and plot $\delta E(t)$ and $\phi(0,t)-\hat{\phi}(x-\Omega t)|_{x=0}$ against t, respectively, in Fig. 3(a) and 3(b). Again it is found that the pinning very effectively stabilizes the unstable state and controls chaos. Two points are worthwhile marking with respect to the pinning. Usually in inhomogeneous cases the efficiency of pinnings depends on the choice of x_0. Thus it is interesting to investigate the influence of pinning position on controlling chaos and to find the most effective pinning point. However, in our model the pinning effect is independent of the choice of x_0 due to the symmetry of Eq. (2). Second, for the x-space pinning there is only a lower threshold for $\lambda(=\lambda_L)$. As $\lambda \geq \lambda_L$ the pinning can always successfully stabilize the aim state. Actually, the larger the λ, the better the controlling. This fact makes the x-space pinning particularly convenient. To our knowledge, our simulation is the first example of using a pinning technique to control chaos. This technique may be used in both theoretical analysis and experimental works.

Bistability is a very important topic in physics, chemistry, biology, and so on. Usually, one gets an S-shaped solution curve, the upper and lower branches are stable and the middle branch unstable. In Fig. 4(a) we take $\Omega=0.56$ and plot E_s against ϵ. Here E_s is the energy of the steady solution of Eq. (2). The dotted line is unstable, the corresponding reference state and energy are denoted by $\hat{\phi}(z)$ and $\hat{E}$, respectively. From the physical point of view, it is very useful to make the middle branch stable by a very weak controlling. Without controlling, in order to drive the system from the lower (upper) branch to the upper (lower) branch one needs large perturbation to overcome the middle barrier. If the middle state is stabilized by a weak feedback we can rather flexibly realize the lower or upper state by taking off the feedback and switching on a small pushing up or down. This idea is realized in our case by using both Eq. (5) and Eq. (6). In Figs. 4(b) and 4(c) we use Eq. (5) and plot $\delta E=E(t)-\hat{E}$ vs t for $\epsilon=0.07$ [the cross in Fig. 4(a)], $i=2$, with $\lambda=0$ and $\lambda=1.0$, respectively. Without controlling the middle branch $\delta E=0$ is unstable [Fig. 4(b)]. With the controlling the system is eventually locked to the middle state though the initial condition is far from the aim state [Fig. 4(c)].

This work is supported by National Natural Science Foundation of China, and National Basic Research Project Nonlinear Science.

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