Trace Formula for Riemann Surfaces with Magnetic Field

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The Maass operator for automorphic forms is interpreted as the Hamiltonian of a charged particle moving in a constant magnetic field on a Riemann surface of constant negative curvature. A careful analysis of the classical trajectories in the presence of the magnetic field shows that the action on a periodic orbit is related to the action on a free periodic geodesic. This identification provides a semiclassical interpretation of the Selberg-Maass trace formula for automorphic forms.

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The free motion of a particle on a two-dimensional surface of constant negative curvature has attracted wide attention among physicists in the last few years. It provides one of the simplest examples of a dynamical system with a classical strongly chaotic behavior. From a quantum point of view, one of the interests of this system is that the semiclassical trace formula of Gutzwiller and Balian and Bloch [1], which is an asymptotic formula relating the *classical* periodic orbits to the *quantum* energy spectrum, coincides with the exact Selberg trace formula [2,3]. However, these formulas have different origins: The Gutzwiller one is based upon semiclassical considerations and the Selberg one upon a group-theoretical analysis [3,4]. The exactness of the semiclassical formula is striking, since it is built from the Van Vleck propagator, which is by no means exact on such surfaces.

It is also possible to investigate the case of a charged particle coupled to an external uniform magnetic field. Recent interest in this problem was raised by its possible relevance to the physics of persistent currents in mesoscopic rings [5] and of the quantum Hall effect [5,6]. From a more formal point of view, this problem can be related to the factorization of determinants of generalized Laplacians and to multiloop calculations in fermionic string theories and random surfaces [7]. As far as quantum chaos is concerned, it is interesting to investigate spectral properties of chaotic Hamiltonians coupled to a magnetic field [8]. In particular, quantum scattering properties can be unaffected by the presence of a magnetic field [9,10].

Generalizations of the Selberg trace formula in the presence of a magnetic field (Selberg-Maass trace formula) are well known in the mathematical literature even though the physical interpretation is obviously lacking. In the Selberg-Maass formula, one considers a generalized Laplacian acting on *automorphic forms of weight m.* It is immediately seen that this operator is identical to the Hamiltonian of a charged particle in the presence of a B field. The weight of the automorphic form is related to the field strength $m=2B$ [4,7].

Quite surprisingly, the Selberg-Maass formula is *almost the same* as in the absence of the magnetic field, up to phase factors which can be interpreted as coming from the effect of isolated vortex lines. This result is amazing, since a magnetic field [11] changes drastically the *classical dynamics* of the system.

The main purpose of this Letter is to solve this paradox by interpreting the Selberg-Maass trace formula as a semiclassical formula. More precisely one shows that the action of a particle on a periodic orbit in the presence of a B field is equal to the free action on a free geodesic orbit, the latter being at constant distance from the former. Results will only be sketched, and details will be published elsewhere.

Riemann surfaces of constant negative curvature are built from the infinite surface of constant negative curvature, as the plane torus is built from the plane. A representation of this surface, the *Poincaré upper half plane* [12], is the complex upper half plane $\mathcal{H}=\{z=x+iy|y>0\}$ with metric $(ds)^2=[(dx)^2+(dy)^2]/y^2$. The geodesic distance d between two points z_i,z_f is given by

$$\cosh d(z_i,z_f)=1+[(x_f-x_i)^2+(y_f-y_i)^2]/2y_iy_f\,.$$

(Note that infinity is either a point of the real axis or the point $y\to\infty$.)

The free geodesics on $\mathcal{H}$ are either half circles or straight lines orthogonal to the real axis $y=0$. The *isometries*, i.e., transformations which preserve the distance, give the symmetry group of $\mathcal{H}$. Direct isometries correspond to the transformations $z\to(az+b)/(cz+d)$ with $a,b,c,d\in R$, which can be represented by real 2×2 matrices with a determinant one, i.e., elements of $\mathrm{SL}(2,R)$. One gets a Riemann surface by taking some discrete group of isometries Γ [whose elements γ are in $\mathrm{SL}(2,R)$], which corresponds to a finite area domain $\mathcal{F}\in\mathcal{H}$ whose sides are identified by elements of the group. Trajectories on this surface are built by following free geodesics of the Poincaré upper half plane on disjoint copies $\gamma\mathcal{F}$ of the tiling of $\mathcal{H}$.

From the Laplace operator $\Delta=y^2(\partial_x^2+\partial_y^2)$ one defines the free quantum Hamiltonian $H=-\frac{1}{4}\Delta$ (in suitable units the mass of the particle is set equal to 2). It acts on wave functions $\psi(z)$ which are invariant under the action of the group Γ; i.e., $\psi(z)=\psi(\gamma z)$ for $\gamma\in\Gamma$.

One can build trace formulas either in the energy or in the time representation, starting, respectively, from the energy Green's function or time propagator. Here we will work in the time formalism, but all results can be

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easily translated to the energy one. On these surfaces, there are two *trace formulas* which coincide. The Gutzwiller trace formula for chaotic systems with unstable periodic orbits relates the *partition function* $\mathrm{Tr}[\exp(-tH)]$ to a sum over classical orbits. It is a semiclassical asymptotic formula in the Planck's constant derived mainly by stationary phase considerations. The Selberg formula is an exact relation valid only for constant negative curvature surfaces that expresses the partition function as a sum over *conjugacy classes* of Γ. It is derived by pure group-theoretical considerations. The Selberg and Gutzwiller formulas can be identified because of the one-to-one correspondence between the hyperbolic conjugacy classes of the group and the classical periodic orbits of the system.

Consider a Riemann surface corresponding to the group Γ. First one has to find the periodic orbits (PO) of the system. One can show that the only way to get a PO is to construct an invariant geodesic under the action of some group element $\gamma \in \Gamma$. To see more precisely which geodesic it is, one can use the frame in which γ is diagonal and acts as a dilatation. The unique invariant geodesic is the imaginary axis of the upper half plane. In this frame one also gets the relation between the trace of γ and the length l of the PO $\mathrm{Tr}(\gamma)=2\cosh(l/2)$. Two invariant geodesics can give rise to the same PO; this redundancy is avoided since there is one and only one PO by *conjugacy class* of the group Γ (we recall that two elements $\gamma_1,\gamma_2 \in \Gamma$ are in the same conjugacy class if and only if there is some γ in Γ such that $\gamma\gamma_1\gamma^{-1}=\gamma_2$).

Having the PO and their lengths, one can turn to the semiclassical analysis leading to the Gutzwiller formula. The classical action on a trajectory spanned at a constant speed $\sqrt{C}$, starting at time τ_i from the initial point $z_i=x_i+iy_i$ and ending at time τ_f at the final point $z_f=x_f+iy_f$, is simply $S(z_i,\tau_i|z_f,\tau_f)=C\tau$ where $\tau=\tau_f-\tau_i$. Since the geodesic distance $d(z_i,z_f)$ between z_i and z_f is $d(z_i,z_f)=\sqrt{C}\tau$, one gets $S(z_i,\tau_i|z_f,\tau_f)=d^2(z_i,z_f)/\tau$.

Then one integrates the semiclassical (Van Vleck) propagator over the whole domain by means of the stationary-phase approximation. The result is a sum of exponential terms, each of them associated with a PO, whose argument is the classical action and whose amplitude is related to the monodromy matrix. In this case, this reproduces the Selberg trace formula [strictly speaking the last term (so-called Weyl term) does not follow from these considerations]

$$\mathrm{Tr}e^{-\tau H}=\sum_{\gamma\mathrm{prim}}\sum_{p=1}^{\infty} e^{-\tau/16}\frac{d}{\sinh(pd/2)}\frac{1}{4\sqrt{\pi\tau}}e^{-p^2d^2/\tau}+\frac{V}{2\pi}e^{-\tau/16}\int_0^\infty (be^{-b^2\tau/4}\tanh\pi b)\,db\,. \qquad (1)$$

Here one considers primitive orbits of length d, which, in the Selberg formalism, correspond to hyperbolic conjugacy classes of group elements whose trace is $2\cosh(d/2)$. All repetitions are indexed by the integer p. The term $\tau/16$ in (1) is a Riemannian correction needed to get the correct semiclassical limit for the free propagator. V is the area of the domain.

One can add a magnetic field on the Poincaré upper half plane. In suitable units, the classical Lagrangian of a charged particle in the presence of the B field in the gauge $A_y=0$ reads $L=(\dot{x}^2+\dot{y}^2)/y^2=B\dot{x}/y$. The corresponding Hamiltonian is $H=-\frac{1}{4}[y^2(\partial_x^2+\partial_y^2)+2iBy\partial_x-B^2]$. The classical and quantum analyses have been performed in [11,13]. Classically, one has two types of trajectories: closed circles which do not intersect the real axis (the particle is trapped) below $E_0=B^2/4+\frac{1}{16}$, and Euclidean circles or straight lines which intercept the real axis with an angle α depending on the magnetic field and the velocity (the particle escapes at infinity) above E_0. At the quantum level, this duality manifests itself in a spectrum with a discrete part (Landau bound states) below E_0 and a continuous part (scattering states) above. Therefore, the net effect of the magnetic field is an additional finite set of discrete Landau levels.

On a Riemann surface the Hamiltonian now acts on automorphic forms of weight $2B$ which transform as [4,5]

$$\psi(\gamma z)=\chi(\gamma)\left(\frac{cz+d}{|cz+d|}\right)^{2B}\psi(z)\,, \qquad (2)$$

where $\chi(\gamma)$ is a set of phase factors known as a multiplier system of Γ; they can be viewed as coming from magnetic flux lines going through the holes of the surface. From the Dirac quantization condition $BV=2\pi m$ (m integer) [5] and the Gauss-Bonnet relation $V=4\pi(g-1)$ (g is the genus) it follows that B must be rational. For noninteger B it is known that $\chi(\gamma)$ cannot be taken as trivial [4]. This implies that in this case the Selberg theory can be applied only if one adds to the system a set of magnetic fluxes corresponding to these nontrivial $\chi(\gamma)$.

The theory of automorphic forms on compact surfaces without elliptic points leads to the Selberg-Maass trace formula [4,7]:

$$\begin{aligned}\mathrm{Tr}e^{-\tau H}=&\frac{V}{4\pi}\sum_{0\le m<B-1/2}(2B-2m-1)e^{-[-m(m+1)+B(2m-1)]\tau/4}\\&+\frac{V}{2\pi}e^{-\tau(1/16+B^2/4)}\int_0^\infty db\frac{be^{-b^2\tau/4}}{\cosh2\pi b+\cos2\pi B}\sinh2\pi b\\&+\sum_{\gamma\mathrm{prim}}\sum_{p=1}^{\infty}\chi(\gamma)e^{-\tau/16}\frac{d}{\sinh(pd/2)}\frac{1}{4\sqrt{\pi\tau}}e^{-(p^2d^2/\tau+B^2\tau/4)}\,,\end{aligned} \qquad (3)$$

where V is the area of the fundamental domain and the last sum is still over hyperbolic conjugacy classes whose elements have trace $2\cosh d/2$. From this formula one infers that the bound state part of the spectrum is not affected by the domain, whereas the continuous part of the spectrum becomes discrete. This is reflected in the classical dynamics where a trapped particle does not feel that the domain is compact, but where some trajectories escaping at infinity on $\mathcal{H}$ become periodic on the Riemann surface. The dynamics is integrable below E_0, as on the upper half plane, but it is strongly chaotic above E_0.

In the sequel, one concentrates on the "chaotic" part of the spectrum (since the "integrable" part of the spectrum yields a finite sum on both sides of the Selberg formula). Apart from the phase factors $\chi(\gamma)$, the last term of (3) is essentially identical to the free case $B=0$ (1). The first question to ask is if the sum over conjugacy classes still corresponds to a sum over PO. As before, one must find for every γ of the group the geodesic invariant by its action. By using the frame in which γ is diagonal, one finds *two* invariant geodesics, namely, the two straight lines intercepting the real axis with the angle α. They can be spanned in only one way, due to the breaking of the time reversal symmetry. This gives a one-to-one correspondence between PO and conjugacy classes, since γ is associated with one geodesic, and γ^{-1} with the other.

This analysis shows that to each conjugacy class of the group one can associate a PO, as in the free case. Nevertheless, this orbit is *distinct from the free one, with a different length* l such that $2\cosh[(l\sin\alpha)/2]=\mathrm{Tr}(\gamma)$. Therefore $l\sin\alpha=d$ in (3).

In order to understand the Selberg-Maass formula from a semiclassical point of view, let us look more carefully at these PO in the frame where γ is diagonal. The positive imaginary axis corresponds to the two periodic orbits of the free case (same geometrical orbit, but spanned in two different directions); the two straight lines with angle α with the x axis correspond to the two periodic orbits of the case with magnetic field [see Fig. 1(a)]. (They are in fact hyperbolic circles.) The PO are spanned between a given point z and its image γz.

Is there some relationship between these two types of orbits? It turns out that there is a suitable projection of

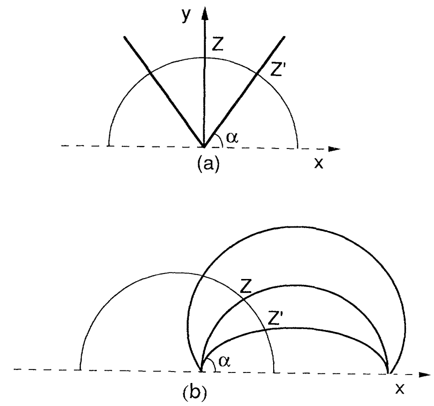

FIG. 1. (a) The invariant geodesics for a diagonal isometry (a dilatation): The y axis is the geodesic for $B=0$; the two oblique lines correspond to the case $B\neq 0$. The mapping $Z\to Z'$ is given by the projection along the circle. (b) The same for an arbitrary isometry: The three thick-lined circles are the three invariant geodesics (the one between is the $B=0$ one); the thin-lined circle gives the mapping.

the two oblique lines which yields exactly the free PO (see Fig. 1). More important, the *action* of the PO with magnetic field only differs from the free action on the projected orbit by a constant shift. The *lengths* of the two orbits are *different*; nevertheless the actions are the same (up to a constant shift).

Let us show how this happens in general. The classical trajectories are parametrized by [one denotes by x',y', $\tau'(x,y,\tau)$ the space-time coordinates on a trajectory in the B field ($B=0$)]

$$x'=x_0+\frac{\sqrt{C}}{K}\frac{B'\sinh\tau'\sqrt{C}}{B+B'\cosh\tau'\sqrt{C}}, \tag{4}$$

$$y'=\frac{2C}{K}\frac{1}{B+B'\cosh\tau'\sqrt{C}}, \tag{5}$$

where $B'=\sqrt{B^2+4C}$. They represent Euclidean circles centered at $[x_0,-B/2K]$, of radius $R'=\sqrt{B^2+4C}/2K$ spanned at constant speed $\sqrt{B^2+4C}/2$ and intersecting the real axis at an angle α, with $\tan\alpha=2\sqrt{C}/B$.

The action of a particle in a B field between the initial point z_i' at time τ_i' and the final point z_f' at time τ_f' is found to be

$$S(z_i',\tau_i'|z_f',\tau_f')=\left(C-\frac{B^2}{4}\right)\tau_f'+B\arcsin\frac{B'+B\cosh\tau_f'\sqrt{C}}{B+B'\cosh\tau_f'\sqrt{C}}-[\tau_f'\to\tau_i']. \tag{6}$$

The points z_i' and z_f' are identified by $z_f'=\gamma z_i'$. In the case of trajectories in a B field one has

$$\gamma=\begin{pmatrix}\cosh\frac{\tau'\sqrt{C}}{2} & \frac{\sqrt{C}}{K}\sinh\frac{\tau'\sqrt{C}}{2}\\ \frac{K}{\sqrt{C}}\sinh\frac{\tau'\sqrt{C}}{2} & \cosh\frac{\tau'\sqrt{C}}{2}\end{pmatrix}. \tag{7}$$

Such an identification has to be accompanied by a gauge transformation. The vector potential is not invariant under the isometries of $\mathcal{H}$. As discussed in [10], invariance is implemented only in a weak sense, namely, $A_\mu(\gamma z')=A_\mu(z')+\partial_\mu^{z'}\Lambda(\gamma,z')$, where $\Lambda(\gamma,z')=2B\arg(cz'+d)$.

In order to compute the action, one has to define carefully the line integral $\int A_\mu (dx^\mu/dt)dt$ on a trajectory. By a choice of suitable transition functions [14], this amounts to canceling the arcsin term in the action (6). One finally obtains the "free" action $S(z_i',\tau_i'|z_f',\tau_f') = (C-B^2/4)\tau'$. One can proceed further by showing that this is the free action on a geodesic spanned at a given distance from the $B\neq 0$ trajectory initially considered. Indeed, consider again the solutions of the equations of motion given above. One has a family of circles indexed by the parameter B with the same points at infinity $z_1 = z_1' = x_0 - \sqrt{C}/K$, $z_2 = z_2' = x_0 + \sqrt{C}/K$. All of them are invariant by the same γ (7). If z is any point on the circle $B=0$, then one can find a mapping $z'(z)$ where z' is on the $B\neq 0$ circle B such that $d(z,z')$ is a constant [see Fig. 1(b)]. It reads

$$z' = \frac{z_2 e^{-i(\pi/2-\alpha)}(z-z_1) - z_1(z-z_2)}{e^{-i(\pi/2-\alpha)}(z-z_1)-(z-z_2)}. \tag{8}$$

One verifies that $\cosh d(z,z') = 1/\sin\alpha$ is indeed constant.

Also, in terms of τ' parametrization ([4] and (5)], this mapping implies $\tau' = \tau$; thus $S(z_i',\tau_i'|z_f',\tau_f') = (C-B^2/4)\tau$. But this action is precisely the free action on the $B=0$ geodesic from point $z_i(z_i')$ at time $\tau_i = \tau_i'$ to the corresponding point $z_f(z_f')$ at time $\tau_f = \tau_f'$.

From this semiclassical analysis it follows that the evolution operator $\mathrm{Tr}(e^{-i\tau H})$ is controlled by an infinite sum of terms of the form $\sum a_p \exp(ip^2d^2/\tau - iB^2\tau)$. Therefore the trace of the heat kernel $\mathrm{Tr}(e^{-\tau H})$ can be readily obtained by the analytic continuation $\tau \to i\tau$. One therefore recovers the free action $p^2d^2/\tau + B^2\tau$ that appears in the trace formula (3).

Monodromy considerations will be identical for both trajectories, since they stay at a constant distance during the time evolution. In fact, orbits with magnetic field are built from free geodesics by attaching a rigid segment to the particle moving on the free orbit. In the trace formula, the amplitude for a given orbit, which measures its *stability*, is obviously not affected by this rigid shift. These considerations imply that the semiclassical analysis for the B field is identical to the free one.

These results also hold for surfaces with elliptic points or infinite "horns" (noncompact surfaces such as Gutzwiller's leaky tori) as far as one is concerned only with the PO sum. They shed some light on the result known for a long time by mathematicians and pointed out in [5]: The "chaotic" spectrum of an *integer B* field [for which one can take $\chi(\gamma)=1$ for all γ] can be obtained from the free spectrum by a shift of $B^2/4$.

There is a conjecture [15] that spectral fluctuations for chaotic time reversal invariant systems obey Gaussian orthogonal ensemble (GOE) statistics, whereas they obey Gaussian unitary ensemble (GUE) statistics when the time reversal symmetry is broken. In the case at hand the *integer* magnetic field shifts the spectrum [4,5]; thus the fluctuations are GOE type, despite an explicit breakdown of time reversal symmetry. This is quite puzzling, since [16] GOE spectral fluctuations with no time reversal symmetry should be connected with *antiunitary symmetries.* Here the only thing one has proven is that *periodic orbits* are degenerate in length. This is usually not sufficient, but we have at our disposal an additional piece of information, namely, that the trace formula is exact. Thus periodic orbits give complete information on the spectrum. This peculiar situation explains the anomalous statistics: The PO doubly degenerate in length mimics the PO of a time reversal invariant system. This can be generalized to any magnetic field provided the character χ verifies $\chi(\gamma) = \chi(\gamma^{-1})$ for all γ in the group.

In conclusion, for a constant negative curvature Riemann surface with quantized magnetic field (i) the Gutzwiller trace formula is exact and coincides with the Selberg-Maass trace formula; (ii) there is a one-to-two correspondence between the *classical* periodic orbits with and without magnetic field. To each free periodic orbit correspond two periodic orbits of the problem with magnetic field, each of them being at constant distance from the free PO; and (iii) this correspondence also explains why there is no transition from GOE to GUE when adding an integer magnetic field.

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