

Nonlocal Symmetry for QED

Martin Lavelle^{1,*} and David McMullan^{2,†}

¹*Institut für Physik, Johannes Gutenberg-Universität, Staudingerweg 7, Postfach 3980,
W-6500 Mainz, Federal Republic of Germany*

²*Dublin Institute for Advanced Studies, School of Theoretical Physics, 10 Burlington Road, Dublin 4, Ireland
(Received 31 March 1993)*

We demonstrate that QED exhibits a previously unobserved noncovariant, nonlocal symmetry. Some consequences are discussed.

PACS numbers: 12.20.-m, 11.30.-j

Quantum electrodynamics (QED) is the cornerstone of modern high energy physics. A generalization of its gauge invariance is found in all other theories of nature. In this Letter we shall show that QED displays a further, quite distinct, symmetry.

In order to quantize QED gauge fixing is essential. Working in Feynman gauge (as we shall throughout this Letter) the Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 + \bar{\psi}(i\not{D} - m)\psi + i\bar{c}\square c, \quad (1)$$

where $D_\mu = \partial_\mu + igA_\mu$ and the ghosts are Hermitian. Although the ghost fields decouple, they are retained in (1) since it is this formulation of QED that can be extended to non-Abelian theories rather than, say, the ghost-free Gupta-Bleuler description.

As is well known the photon has 2, transverse, degrees of freedom. However, in the covariant formulation (1) all four components of the gauge field are present. One can argue that the gauge fixing term has accounted for 1 degree of freedom, while the ghosts heuristically contribute 1 negative degree of freedom. Thus the full Lagrangian (1) does indeed describe the interaction of fermions with the 2 degrees of freedom of the photon. This line of argument can be put on a firmer footing by exploiting the Becchi-Rouet-Storg-Tyutin (BRST) invariance of the QED Lagrangian.

The BRST transformations are [1]

$$\begin{aligned} \delta A_\mu &= \partial_\mu c, & \delta c &= 0, & \delta \bar{c} &= -i\partial_\mu A^\mu, \\ \delta \psi &= -igc\psi, & \delta \bar{\psi} &= -ig\bar{\psi}c, \end{aligned} \quad (2)$$

and we note that the BRST transform of a field has increased its ghost number by 1. The invariance of (1) under these transformations can be shown in three steps: The original photonic and Dirac Lagrangians are each separately invariant (reflecting the close connection between gauge invariance and BRST invariance), while the gauge fixing and ghost terms together are invariant. One of the most remarkable properties of the BRST transformation is that it vanishes when applied twice to any field; $\delta^2 \equiv 0$. This nilpotency property can be seen from (2) us-

ing the (uncoupled) ghost equation of motion. (This reliance on the equations of motion can be removed by introducing auxiliary fields as in Ref. [1].)

The BRST invariance of the Lagrangian (1) allows for a succinct characterization of the physical states of the theory. The transformations (2) are generated by the conserved, Hermitian charge

$$Q := \int d^3x [-\dot{\pi}_0(x)c(x) + \pi_0(x)\dot{\bar{c}}(x)], \quad (3)$$

where $\pi_0 := -\partial_\mu A^\mu$. The physical states are then identified with those states, $|\psi\rangle$, which satisfy $Q|\psi\rangle = 0$. This is not the whole story, though, since the nilpotency of the BRST transformation implies that $Q^2 = 0$; hence any state of the form $|\psi\rangle = Q|\chi\rangle$, for any $|\chi\rangle$, will trivially be physical in this sense. More properly, then, the physical states should be identified with the quotient of the BRST invariant (closed) states with these trivial (exact) states. Working on the Fock space for this theory constructed out of the free fields (which are identified with the asymptotic in and out states), it was shown in Ref. [1] that the photonic sector of the physical states could be identified with the states built from the transverse components of the photon. Even in this Abelian theory this is quite an involved analysis.

This Fock space discussion is only strictly relevant to the free theory: When matter is present the infrared structure of the theory [2] implies that asymptotic states which have an arbitrary number of photons in them must be allowed. The direct extension of the arguments presented in Ref. [1] to such coherent states is then not clear.

This use of the BRST charge to characterize the physical states is familiar in mathematics and would be called a cohomology theory—the physical states then being the zeroth cohomology of the appropriate complex. Thus the BRST charge is playing a role similar to the exterior derivative, d , acting on differential forms. Mathematicians have developed many techniques for analyzing such structures (see, for example, Ref. [3]), the most powerful of which is to introduce an adjoint operation (denoted by d^* for the differential forms example). This is also nilpotent and can be used to refine the description of the d closed forms. It would be useful to have a similar adjoint

to the BRST charge; however, it is not clear that such an object can be constructed since, in addition to its various algebraic properties, it must also be conserved if it is to have any physical significance.

The anti-BRST transformation [1], δ_{anti} , is an example of a type of adjoint to the BRST transformation. Its existence relies on the simple fact that the Lagrangian (1) is invariant under the interchange $c \rightarrow i\bar{c}$ and $\bar{c} \rightarrow ic$. Acting on the fields δ_{anti} essentially reproduces (2) but with the ghost and antighost interchanged. Clearly this transformation has the property that it reduces the ghost number of a state by 1. However, its close connection with the BRST charge, and hence the gauge invariance of the theory, means that it has few of the useful properties desired from an adjoint. In particular, it anticommutes with the BRST charge, $\delta\delta_{\text{anti}} + \delta_{\text{anti}}\delta = 0$; thus there is no analog of the Laplacian $dd^* + d^*d$ and the related harmonic description of cohomology found in differential geometry.

In Ref. [4] it was argued that, for simple quantum mechanical systems, the appropriate concept of an adjoint (or more properly, a dual) to the BRST transformation is not one based on the existence of pairing between states (as in the relationship between d and d^*), but rather it is a transformation that is compatible with the gauge fixing conditions. An extension of that argument to QED would suggest that we are looking for a symmetry transformation δ^\perp of (1) such that, as well as decreasing the ghost number of the fields by 1, acting on the gauge fixing condition it gives 0:

$$\delta^\perp(\partial_\mu A^\mu) \equiv 0. \tag{4}$$

(In a formulation with the auxiliary field π_0 , this condition could be replaced with the weaker condition that $\pi_0 + \partial_\mu A^\mu$ is invariant.) We say that relation (4) is dual to the BRST invariance of gauge invariant quantities, i.e., dual to the relations $\delta(\frac{1}{4}F^{\mu\nu}F_{\mu\nu}) = 0$ and $\delta(\bar{\psi}(i\not{D} - m)\psi) = 0$.

The simplest way to ensure (4) is to take $\delta^\perp A_0 = i\bar{c}$ and $\delta^\perp A_i = i(\partial_i \partial_0 / \nabla^2)\bar{c}$, where $\nabla^2 = \partial_i \partial_i$. So we see that solving (4) forces us to abandon covariance and locality. The lack of covariance in this transformation should not come as too much of a surprise since we know that the two transverse physical photonic states cannot be expressed covariantly. Extending this transformation to the other fields is far from unique; however, requiring that we have a symmetry of (1) allows us to finally arrive at the set of transformations:

$$\begin{aligned} \delta^\perp A_0 &= i\bar{c}, \quad \delta^\perp A_i = i \frac{\partial_i \partial_0}{\nabla^2} \bar{c}, \\ \delta^\perp c &= A_0 - \frac{\partial_i \partial_0}{\nabla^2} A_i + \frac{g}{\nabla^2} J_0, \quad \delta^\perp \bar{c} = 0, \\ \delta^\perp \psi &= \left[\frac{g}{\nabla^2} \partial_0 \bar{c} \right] \psi, \quad \delta^\perp \bar{\psi} = \bar{\psi} \frac{g}{\nabla^2} \partial_0 \bar{c}, \end{aligned} \tag{5}$$

where J_0 is the current density $\bar{\psi}\gamma_0\psi$. Just as was the case for the BRST transformations this transformation can be seen to be nilpotent, i.e., $(\delta^\perp)^2 \equiv 0$, with the aid of the antighost equation of motion. (Again this dependence may be avoided by introducing an auxiliary field.)

It is important to clearly spell out why this is a symmetry of the Lagrangian (1). By construction, it is now the gauge fixing term that is invariant while the photonic and Dirac parts of the Lagrangian need, in addition, the ghost term to be invariant. Under the transformation (5) the Lagrangian (1) changes by a total divergence

$$\delta^\perp \mathcal{L} = \partial_\mu \Lambda^\mu, \tag{6}$$

where

$$\begin{aligned} \Lambda^0 &= i\partial^0 \bar{c} \delta^\perp c - i\bar{c} \partial^0 \delta^\perp c, \\ \Lambda^i &= i\partial^i \bar{c} \delta^\perp c - i\bar{c} \partial^i \delta^\perp c - i\partial^i \bar{c} \left[\frac{\partial_0 A^i}{\nabla^2} - \frac{\partial^i A_0}{\nabla^2} \right]. \end{aligned} \tag{7}$$

Note that, due to the presence of the nonlocal, $1/\nabla^2$, terms in Λ^μ , we cannot immediately deduce from (6) that this is a symmetry. This is because, in general, any transformation of a Lagrangian can be written as a total divergence if we allow nonlocal terms; i.e., we may always write an arbitrary function F as $F = \partial_\mu [(\partial^\mu / \square)F]$. The fundamental quantity in the quantum theory is the action; thus we must now check that our transformation indeed preserves the action.

Recall that we want to construct the action from the Lagrangian (1) where

$$S = \int_{T_1}^{T_2} dt \int_{B_r} d^3x \mathcal{L}, \tag{8}$$

and we are calculating the action between two times $[T_1, T_2]$ and, for the moment, we have a spatial ball, B_r , of radius r . We will want to take the $r \rightarrow \infty$ limit, but we will not need to take the $[T_1, T_2] \rightarrow [-\infty, \infty]$ limit. Under the change (5) we get

$$S \rightarrow S + \Lambda(T_2) - \Lambda(T_1) + \int_{T_1}^{T_2} dt \int_{B_r} d^3x \partial_i \Lambda^i, \tag{9}$$

where

$$\Lambda(T) = \int_{B_r} d^3x \Lambda^0(T, x). \tag{10}$$

The $\Lambda(T)$ terms now depend on the whole of the space slices at the end points in time; however, this will clearly not affect the dynamics between the initial and final times. The last term is potentially dangerous since it will alter the action between the end points, and thus could alter the dynamics. We use Stokes' theorem to write

$$\int_{B_r} d^3x \partial_i \Lambda^i = \int_{S^2} d^2\sigma_i \Lambda^i. \tag{11}$$

If the Λ^i 's were local this would tend to zero as $r \rightarrow \infty$ since the fields fall off to zero at infinity. But since the Λ^i 's are nonlocal, then even as $r \rightarrow \infty$, they will receive

contributions from the whole of the spatial slice—so (11) might not vanish. However, we are dealing with terms of the form

$$\int_{S^2} d\sigma^i \frac{f(x)}{\nabla^2} = -\frac{1}{4\pi} \int_{S^2} d\sigma^i \int d^3y \frac{f(y)}{|x-y|}. \quad (12)$$

Now as $r \rightarrow \infty$, i.e., as $x \rightarrow \infty$, for finite y we have $1/|x-y| \rightarrow 0$. This is not the case, though, if y and x are close. But then, for large x , $f(y) \rightarrow 0$ since we demand good boundary conditions on the fields. We conclude that even though our symmetry is nonlocal, it is indeed a symmetry in the nontrivial sense.

Before briefly discussing some of the consequences of this new symmetry, we note that its actions on the gauge fields and fermions can formally be derived from the BRST transformations (2) under the substitution $c \rightarrow i\bar{c}/\partial_0$ and by using the equations of motion. The action on the ghosts then follows from the required invariance of the Lagrangian. It is not clear to us if this trick sheds any light on this new symmetry. Indeed the use of the classical equations of motion in a transformation does not guarantee the invariance of the action. For example, if the classical equations of motion are used then we can also replace $\delta^+ c$ in (5) by

$$\delta_a^+ c = A_0 - a \frac{\partial_i \partial_0}{\nabla^2} A_i + \frac{g}{\square} \left[a \frac{\partial_0 \partial_0}{\nabla^2} - 1 \right] J_0. \quad (13)$$

For all a this transformation will preserve the classical equations of motion. However, the Lagrangian only transforms into a total divergence *without use of the classical equations of motion* for the specific choice $a = 1$.

We now wish to analyze some consequences of this symmetry. Given that this is a symmetry we can use it, in much the same way as BRST is used, to generate Ward type identities. So consider the identity

$$\langle T[A_0(x)c(y)] \rangle \equiv 0. \quad (14)$$

Applying δ^+ to this yields

$$-iD(q) + D_{00}(q) - \frac{qi q_0}{q^2} D_{i0}(q) - \frac{g}{q^2} \Lambda_{00}(q) = 0, \quad (15)$$

where

$$\begin{aligned} D(q) &:= \int d^4x e^{iqx} \langle T[c(x)\bar{c}(0)] \rangle, \\ D_{\mu\nu}(q) &:= \int d^4x e^{iqx} \langle T[A_\mu(x)A_\nu(0)] \rangle, \\ \Lambda_{00}(q) &:= \int d^4x e^{iqx} \langle T[A_0(x)J_0(0)] \rangle. \end{aligned} \quad (16)$$

This is not one of the usual covariant Ward identities. However, it is straightforward to check that it is indeed fulfilled in QED in the Feynman gauge. The noncovariance of this identity is a consequence of the noncovariance of the symmetry and of the noncovariance of the physical fields. A fuller account of the identities derivable from this new symmetry will be presented elsewhere.

Returning now to our original motivation for searching for this symmetry, we recall that physical states have been characterized as being BRST invariant. To refine this description we now impose an additional condition that physical fields must also be invariant under the symmetry (5). Thus the physical states of the theory must satisfy the conditions

$$Q|\psi\rangle = Q^+|\psi\rangle = 0, \quad (17)$$

where the anti-Hermitian charge Q^+ , which generates the transformations (5), is given by

$$Q^+ := \int d^3x \left[-i \frac{\pi_0(x)}{\nabla^2} \dot{\bar{c}}(x) + i\pi_0(x)\bar{c}(x) \right]. \quad (18)$$

Combining these requirements we see that a physical state will, in addition, satisfy

$$\mathcal{N}|\text{phys}\rangle = 0, \quad (19)$$

where the Hermitian Laplacian type operator \mathcal{N} is given by

$$\mathcal{N} := QQ^+ + Q^+Q. \quad (20)$$

Although this does not have all the properties one might wish from a Laplacian, in that $\mathcal{N}c = \mathcal{N}\bar{c} = 0$, it does impose additional restrictions on the photonic states and one can show [5] that in the coherent space approach, the physical states are built up from the transverse components of the photon and the fields

$$\begin{aligned} \psi_{\text{phys}}(x) &= \exp \left[-ig \frac{\partial_i A^i(x)}{\nabla^2} \right] \psi(x), \\ \bar{\psi}_{\text{phys}}(x) &= \exp \left[ig \frac{\partial_i A^i(x)}{\nabla^2} \right] \bar{\psi}(x), \end{aligned} \quad (21)$$

which are Dirac's physical electrons [6].

In summary, we have found a new symmetry for QED. This symmetry is noncovariant and nonlocal. Evidently just as BRST has a partner in anti-BRST symmetry, where ghosts and antighosts are interchanged, an "anti-inversion" of this symmetry is easily constructed. The δ^+ symmetry may be used to refine the characterization of physical states given by the BRST charge. We have also shown that this new symmetry will generate new Ward type identities in the quantum theory. The extension of this symmetry to other gauges, including noncovariant ones, and the geometric role of this symmetry will be presented elsewhere.

M.L. thanks the Dublin Institute for Advanced Studies for their hospitality and the Graduierten Kolleg of Mainz University for support. D.M. thanks the University of Mainz for hospitality.

*Electronic address: lavelle@vipmza.physik.uni-mainz.de

- †Permanent address: School of Mathematics and Statistics, University of Plymouth, Drake Circus, Plymouth, PL4 8AA, United Kingdom. Electronic address: d.mcmullan@plymouth.ac.uk
- [1] N. Nakanishi and I. Ojimi, *Covariant Operator Formalism of Gauge Theories and Quantum Gravity* (World Scientific, Singapore, 1990).
- [2] T. W. B. Kibble, in *Mathematical Methods in Theoretical Physics*, edited by K. T. Mahanthappa and W. E. Brittin (Gordon and Breach, New York, 1969).
- [3] R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology* (Springer-Verlag, New York, 1982).
- [4] D. McMullan, *Commun. Math. Phys.* **149**, 161 (1992).
- [5] M. Lavelle and D. McMullan, "Physical States and Greens Functions in Gauge Theories: I QED" (to be published).
- [6] P. A. M. Dirac, *Principles of Quantum Mechanics* (Oxford Univ. Press, Oxford, 1958), p. 302.