Wess-Zumino-Witten Model Based on a Nonsemisimple Group

Chiara R. Nappi and Edward Witten

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, New Jersey 08540

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We present a conformal field theory which describes a homogeneous four dimensional Lorentz-signature space-time. The model is an ungauged Wess-Zumino-Witten model based on a central extension of the Poincaré algebra. The central charge of this theory is exactly four, just like four dimensional Minkowski space. The model can be interpreted as a four dimensional monochromatic plane wave. As there are three commuting isometries, other interesting geometries are expected to emerge via O(3,3) duality.

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In this paper, we will describe a new example of a homogeneous (but not isotropic) four dimensional space-time—with Lorentz signature—that can be constructed from an ungauged Wess-Zumino-Witten (WZW) model. The model has c (or in the supersymmetric case $\hat{c}$) equal to 4, so it can be directly substituted for four dimensional Minkowski space and regarded as a solution of any more or less realistic string theory. It describes a special case of a monochromatic plane wave with more than the usual symmetry.

The model is a WZW model based on a certain nonsemisimple Lie algebra of dimension four. The algebra has the following explicit description:

$$[J,P_i]=\epsilon_{ij}P_j,\quad [P_i,P_j]=\epsilon_{ij}T,\quad [T,J]=[T,P_a]=0\,. \tag{1}$$

This algebra is a central extension of the 2D Poincaré algebra to which it reduces if one sets $T=0$. We will call the corresponding simply connected group R.

In general, given a Lie algebra with generators T_A (here $T_A=P_1,P_2,J,T$) and structure constants f_{AB}^D (so $[T_A,T_B]=f_{AB}^D T_D$) to define a WZW model, one needs a bilinear form Ω_{AB} in the generators T_A, which is symmetric ($\Omega_{AB}=\Omega_{BA}$), invariant,

$$f_{AB}^D\Omega_{CD}+f_{AC}^D\Omega_{BD}=0\,, \tag{2}$$

and nondegenerate (so that there is an inverse matrix Ω^{AB} obeying $\Omega^{AB}\Omega_{BC}=\delta_C^A$). (Usually, Ω must obey a certain integrality condition, but that will have no analog here because of the simple structure of R. That will be clear from the ability below to reduce the Wess-Zumino term to an integral over Σ, without introducing any singularities.) Then the WZW action on a Riemann surface Σ is

$$S_{\rm WZW}(g)=\frac{1}{4\pi}\int_\Sigma d^2\sigma\,\Omega_{AB}A_\alpha^A A^{B\alpha}+\frac{i}{12\pi}\int_B d^3\sigma\,\epsilon_{\alpha\beta\gamma}A^{A\alpha}A^{B\beta}A^{C\gamma}\Omega_{CD}f_{AB}^D\,, \tag{3}$$

where the A_α^A's are defined via $g^{-1}\partial_\alpha g=A_\alpha^A T_A$. Here B is a three-manifold with boundary $\partial B=\Sigma$, and g is a map of Σ to R (extended in an arbitrary fashion to a map from B). Usually for semisimple groups one can choose the bilinear form $\Omega_{AB}=f_{AC}^D f_{BD}^C$ (which is equivalent to $\mathrm{Tr}T_A T_A$ with the trace taken in the adjoint representation). However, for nonsemisimple groups this quadratic form is degenerate, and indeed for the algebra (1) one gets

$$\Omega_{AB}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&-2&0\\0&0&0&0\end{pmatrix}. \tag{4}$$

Nevertheless, the R Lie algebra does have another nondegenerate bilinear form [1], i.e.,

$$\Omega_{AB}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}. \tag{5}$$

It is easily shown that the most general invariant quadratic form on this Lie algebra is a linear combination of these:

$$\Omega_{AB}=\begin{pmatrix}k&0&0&0\\0&k&0&0\\0&0&b'&k\\0&0&k&0\end{pmatrix}. \tag{6}$$

By setting $b=b'/k$ we can write Ω_{AB} and its inverse as

$$\Omega_{AB}=k\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&b&1\\0&0&1&0\end{pmatrix},\quad \Omega^{-1}=\Omega^{AB}=\frac{1}{k}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&-b\end{pmatrix}. \tag{7}$$

This metric on the Lie algebra has signature $+++-$, and that will therefore be the signature of the space-time described by the corresponding WZW model.

In order to write (3) explicitly we need to find the A_A's. To this purpose we use the following parametriza-

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tion of the group manifold:

$$g=e^{\sum_i a_i P_i} e^{uJ+vT} . \tag{8}$$

(In fact by using the commutation relations to move all of the J's and T's to the right, any group element can be uniquely brought to this form; this is somewhat like the process of normal ordering of an exponential of a sum of creation and annihilation operators.) By using

$$e^{uJ}P_i e^{-uJ}=\cos u\, P_i+\epsilon_{ij}\sin u\, P_j \tag{9}$$

and

$$\partial_a e^H=\int_0^1 dx\, e^{xH}\partial_a H\, e^{(1-x)H}$$

we compute

$$\partial_a g=e^{\sum_i a_i P_i}(\partial_a a_i\, P_i+\tfrac{1}{2}\epsilon_{jk}T\,\partial_a a_j\, a_k)e^{uJ+vT}+g\partial_a(uJ+vT) . \tag{10}$$

By using (10) and (9) we find

$$g^{-1}\partial_a g=(\cos u\,\partial_a a_k+\sin u\,\epsilon_{jk}\,\partial_a a_j)P_k+(\partial_a u)J+(\partial_a v+\tfrac{1}{2}\epsilon_{jk}\,\partial_a a_j\, a_k)T , \tag{11}$$

from which we can read off the A^{A}'s and compute

$$A^A_a A^{Ba}\Omega_{AB}=\partial_a a_k\,\partial^a a_k+2\partial_a v\partial^a u+b\,\partial_a\partial^a u+\epsilon_{jk}\,\partial^a u\,\partial_a a_j\, a_k , \tag{12}$$

$$\epsilon_{\alpha\beta\gamma}A^{A\alpha}A^{B\beta}A^{C\gamma}\Omega_{CD}f^D_{AB}=2\epsilon_{\alpha\beta\gamma}\times 3\partial^\alpha[u\partial^\beta a_1\,\partial^\gamma a_2] . \tag{13}$$

Finally, the Lagrangian looks like

$$\frac{k}{4\pi}\int d^2\sigma[\partial_a a_k\,\partial_a a_k+2\partial_a v\,\partial^a u+b\,\partial_a u\,\partial^a u+\epsilon_{jk}\,\partial^a u\,\partial_a a_j\, a_k+i2\epsilon_{\beta\gamma}u\,\partial^\beta a_1\,\partial^\gamma a_2] . \tag{14}$$

By identifying this Lagrangian with the σ-model action of the form

$$S=\int d^2\sigma(G_{MN}\,\partial_a X^M\,\partial^a X^N+iB_{MN}\epsilon_{\alpha\beta}\partial^\alpha X^M\,\partial^\beta X^N) , \tag{15}$$

where $X^M=(a_1,a_2,u,v)$, one can read off the background space-time metric and antisymmetric tensor field. The space-time geometry is described by a Lorentz-signature metric G_{MN},

$$G_{MN}=\begin{pmatrix} 1 & 0 & \frac{a_2}{2} & 0\\ 0 & 1 & -\frac{a_1}{2} & 0\\ \frac{a_2}{2} & -\frac{a_1}{2} & b & 1\\ 0 & 0 & 1 & 0\end{pmatrix},\quad G^{-1}=G^{MN}=\begin{pmatrix} 1 & 0 & 0 & -\frac{a_2}{2}\\ 0 & 1 & 0 & \frac{a_1}{2}\\ 0 & 0 & 0 & 1\\ -\frac{a_2}{2} & \frac{a_1}{2} & 1 & \frac{a_1^2}{4}+\frac{a_2^2}{4}-b\end{pmatrix}, \tag{16}$$

and $B_{12}=u$, and of course the dilaton is constant because of the homogeneity of the group manifold. (In gauged WZW models, the nonconstant dilaton emerges from integration over the gauge fields.) The corresponding space-time metric is

$$ds^2=da_k\,da_k+2du\,dv+\epsilon_{jk}\,da_j\,a_k\,du+b\,du^2 . \tag{17}$$

As a WZW model, this model should be conformally invariant. To check this, we first look at the one-loop beta function equations

$$R_{MN}-\tfrac{1}{4}H^2_{MN}-D_M D_N\phi=0 ,$$
$$D^L H_{LMN}+D^L\phi H_{LMN}=0 , \tag{18}$$
$$-R+\tfrac{1}{12}H^2+2\Delta^2\phi+(\Delta\phi)^2+\Lambda=0 ,$$

where $H_{LMN}=D_{[L}B_{MN]}$ and $H^2_{MN}=H_{MPR}H_N^{PR}$, $H^2=H_{MPR}H^{MPR}$, and $\Lambda=2(c-4)/3\alpha'$. One quickly finds that the only nonzero component of R_{MN} is $R_{33}=\frac{1}{2}$, and as the only nonzero component of B is $B_{12}=u$, the only nonzero component of H is $H_{312}=1$. As $G^{33}=0$, $R=H^2=0$. The only nonzero component of H^2_{MN} is $H^2_{33}=2$. Putting these pieces together, one verifies Eqs. (18) with $\Lambda=0$ and $c=4$.

A more direct way to see that the one-loop beta function equations are satisfied is to actually compute the one-loop diagrams that contribute to the beta functions. First of all, the only one-loop diagram that one can draw is the one with two u's in the external lines, and a_i's in the internal lines. This is because there is no uu propagator, as $G^{33}=0$. Moreover, by comparing the interaction terms in the Lagrangian one notices that they give opposite contributions to this one-loop diagram. Indeed the interaction term contained in the space-time metric piece of the Lagrangian is

$$\epsilon_{jk}a_k\,\partial_a a_j\,\partial^a u=\eta_{\beta\gamma}\epsilon_{jk}a_k\,\partial^\beta a_j\,\partial^\gamma u , \tag{19}$$

where $\eta_{\beta\gamma}$ is the world-sheet metric. The interaction term associated with the antisymmetric vector field is

$$2iu\epsilon_{\beta\gamma}\partial^\beta a_1\,\partial^\gamma a_2=-i2\epsilon_{\beta\gamma}a_1\,\partial^\gamma a_2\,\partial^\beta u=i\epsilon_{\beta\gamma}\epsilon_{jk}a_k\,\partial^\gamma a_k\,\partial^\beta u . \tag{20}$$

The above couplings differ only because one has $\eta_{\beta\gamma}$ and the other has $i\epsilon_{\beta\gamma}$. They give opposite contributions to the one-loop diagrams since

$$\eta^{\alpha}{}_{\beta}\eta_{\alpha\gamma}=\eta_{\beta\gamma},\quad i^2\epsilon^{\alpha}{}_{\beta}\epsilon_{\alpha\gamma}=-\eta_{\beta\gamma}. \tag{21}$$

This is exactly what happens in [2,3]. Moreover there are no higher loop contributions to the beta function. Indeed one can check that no higher loop diagrams can even be drawn, because all of the interaction vertices have an external u, but the uu propagator vanishes. Therefore $c=4$ identically in perturbation theory.

The statement can be confirmed nonperturbatively by generalizing the Sugawara construction to nonsemisimple algebras. One way is to follow the approach in [4] and check that the tensor

$$L^{AB}=\frac{1}{2}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&1-b\end{pmatrix} \tag{22}$$

satisfies the Virasoro master equation

$$L^{AB}=2L^{AC}\Omega_{CD}L^{DB}-L^{CD}L^{EF}f^{A}_{CE}f^{B}_{DF}-L^{CD}f^{F}_{CE}f^{(A}_{DF}L^{B)E}. \tag{23}$$

This means that one can carry out the Sugawara construction and that the central charge is $c=2\Omega_{AB}L^{AB}=4$. A similar analysis has been carried out for a larger class of nonsemisimple groups [5].

The metric (16) can be turned into a more familiar form by using the following representation for g:

$$g=e^{aP_1}e^{uJ}e^{fP_1+vT}, \tag{24}$$

which exhibits the three commuting symmetries of the model (the left and right action of P_1 and the left or right action of the central object T all commute). This representation (24) can be transformed into the form (8) used earlier via

$$a_1\to a+f\cos u,\quad a_2\to f\sin u\,,$$
$$v\to v+\tfrac{1}{2}af\sin u\,,\quad u\to u\,. \tag{25}$$

Then the metric (17) turns into

$$ds^2=da^2+df^2+2da\,df\cos u+2du\,dv+b\,du^2. \tag{26}$$

The above metric exhibits three commuting symmetries which are realized as translations

$$a\to a+c_1,\quad f\to f+c_2,\quad v\to v+c_3 \tag{27}$$

for arbitrary constants c_1,c_2,c_3. Obviously, translation in v is a null vector. The above metric can be recognized as the metric [6] of a plane wave. This plane wave is monochromatic; it is an extremely strong wave, since there are actually in this coordinate system two singularities in each period—when $\cos u=\pm 1$. It is worthwhile to notice that both in (17) and in (26), the parameter b can be absorbed into a shift of v, i.e., $v\to v=bu/2$.

Of course, the model really has more symmetry than is manifest in either of the two ways of writing the metric given above. From its origin as a WZW theory, the model clearly describes a homogeneous space-time; the left and right action of R on itself gives *in toto* a seven-dimensional symmetry group of the space-time. (R is four dimensional, but as the left and right actions of the central generator T coincide, the total number of symmetries coming from the left and right action of R on itself is seven rather than eight.) The extra symmetries determine the particular strong field nature of this plane wave.

Our explicit argument showing that the higher order corrections to the beta function vanish because there simply are not any suitable Feynman diagrams is probably relevant to a larger class of plane waves [7].

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