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Exact Results for the One-Dimensional Self-Organized Critical Forest-Fire Model

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We present the analytic solution of the self-organized critical (SOC) forest-fire model in one dimension proving SOC in systems without conservation laws by analytic means. Under the condition that the system is in the steady state and very close to the critical point, we calculate the probability that a string of n neighboring sites is occupied by a given configuration of trees. The critical exponent describing the size distribution of forest clusters is exactly $\tau = 2$ and does not change under certain changes of the model rules. Computer simulations confirm the analytic results.

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The concept of self-organized criticality (SOC) [1] has attracted much attention during the last few years since it might explain the origin of fractal structures and of $1/f$ noise in nature. While the sandpile model [1], which is the prototype for SOC, is well understood and has also been investigated analytically [2], the investigation of other SOC systems is mainly restricted to computer simulations [3–6]. In particular, the mechanism leading to SOC in nonconservative systems is barely understood. So far, there exists no proof and it has even been questioned whether such nonconservative systems can become critical.

A recent paper introduced a critical forest-fire model which is nonconservative and showed that a double separation of time scales leads to SOC [7]. A series of computer simulations confirm the criticality of the model [8–11], and a scaling theory yields relations between various critical exponents [10,11]. An analytic proof for the criticality of this model, however, has not been given so far. Reference [9] contains the mean-field theory of the forest-fire model, which seems to agree with simulations in high dimensions. Critical exponents for the one-dimensional forest-fire model have been derived by another mean-field approximation in [12], but do not agree with simulations.

In this paper, we present the analytic solution of the one-dimensional forest-fire model, thus proving for the first time that a nonconservative system can indeed show SOC. The analytic calculation shows also that the value of the critical exponent which characterizes the size distribution of forest clusters does not change under certain changes of the model rules. The values of the critical exponents have already been given in [7] by an argument which was rather intuitive and failed in higher dimensions.

The forest-fire model [7] is defined on a d-dimensional hypercubic lattice with L^d sites. Each site is occupied by either a tree or a burning tree, or is empty. The state of the system is parallely updated by the following rules: (i) A burning tree becomes an empty site. (ii) A green tree becomes a burning tree if at least one of its nearest neighbors is burning. (iii) A green tree becomes a burning tree with probability $f \ll 1$ if no neighbor is burning. (iv) At an empty site a tree grows with probability p.

A forest-fire model without rule (iii) was introduced earlier [13].

After a transition period, the model assumes a steady state with a constant mean forest density ρ. Then the mean number of growing trees equals the mean number of burning trees. During one time step, there are on an average $f\rho L^d$ lightning strokes in the system, and $p(1-\rho)L^d$ trees grow. Consequently, the mean number of trees destroyed by a lightning stroke is

$$\bar{s} = (f/p)^{-1}(1-\rho)/\rho. \qquad (1)$$

When the tree growth parameter p is small enough for a given value f/p, forest clusters that are struck by lightning burn down before new trees grow at their edge. In this case the dynamics of the system are completely determined by the ratio f/p. In the limit $f/p \to 0$, $\bar{s}$ diverges, and the system approaches a critical point where the size distribution of forest clusters $n(s)$ obeys a power law. Since any finite system is never exactly at the critical point, the power law breaks down at a cluster size $s_{\max} \propto (f/p)^{-\lambda}$, and is given by $n(s) \propto s^{-\tau}\mathcal{C}(s/s_{\max})$. The cutoff function $\mathcal{C}$ is $\mathcal{C}(x) = 1$ for $x < 1$ and $\mathcal{C}(x) \to 0$ for large x. The critical behavior in the forest-fire model can be observed independently of the initial conditions and over a wide range of parameter values, as long as $f \ll p$ with sufficiently small p, i.e., as long as time scales are separated. Then the forest-fire model is *self-organized critical* [1].

We now turn to the one-dimensional case of this model and investigate the properties of its steady state. To avoid finite-size effects, we always assume that the system size L is $L \gg p/f$. The condition of time-scale separation reads in one dimension $p \ll f/p$. Consequently, there are never two lightning strokes or two growing trees at the same time within a distance $\lesssim p/f$. Let $n(s)$ be the mean number of forest clusters of s trees, divided by the number of sites L. In [7], we already derived the following relations:

$$\sum_{s=1}^{\infty} n(s) = (1-\rho-\rho f/p)/2,$$
$$\sum_{s=1}^{\infty} s n(s) = \rho, \qquad (2)$$
$$\sum_{s=1}^{\infty} s^2 n(s) = (1-\rho)p/f.$$

The first relation follows from the condition that the total number of forest clusters is constant in the steady state.

Before continuing, we need some definitions. Each site can be in two possible states which we denote by 0 (empty site) and 1 (tree). Let $i_\alpha \in \{0,1\}$ be the state of site α and let $P_n(i_1,\ldots,i_n)$ be the probability that an arbitrarily chosen string of n sites is in the state $(i_1,\ldots,i_n)$. We also define the conditional probabilities $f_n(i_1,\ldots,i_n)$ that the configuration $(i_1,\ldots,i_n)$ is ignited by fire during one time step. $f_n(i_1,\ldots,i_n)$ is the sum of the probability that a tree belonging to the configuration $(i_1,\ldots,i_n)$ is struck by lightning, and, if $i_1 = 1$ or $i_n = 1$, the probability that fire enters the configuration from outside. In the following, we calculate the probabilities P_n and f_n, neglecting terms which are smaller by a factor f/p than the leading terms. The results become more and more exact when the critical point is approached.

The probabilities $P_1(i)$ can be expressed in terms of the mean forest density: $P_1(1) = \rho$ and $P_1(0) = 1-\rho$. At any given site, growth and burning of trees occur equally often, and therefore

$$f_1(1) = p(1-\rho)/\rho. \qquad (3)$$

$f_1(1)$ is the sum of the probability that a tree is ignited by a neighbor and the probability f that it is struck by lightning. The latter is smaller by a factor f/p than $f_1(1)$ and therefore negligible.

In addition to symmetry relations, the probabilities $P_2(i_1,i_2)$ satisfy $P_2(00) + P_2(10) = 1-\rho$ and $P_2(10) + P_2(11) = \rho$. The probabilities $f_2(i_1,i_2)$ are related by $f_2(10)P_2(10) + f_2(11)P_2(11) = f_1(1)P_1(1) = p(1-\rho)$ and, since the system is in the steady state, by $f_2(11)P_2(11) = 2pP_2(10)$ and $[p + f_2(10)]P_2(10) = pP_2(00)$. When a tree is ignited, the fire comes either from the right neighbor or from the left neighbor. This leads to $p(1-\rho) = f_2(11)P_2(11)$ and $f_2(10) = 0$, and finally to

$$P_2(00) = P_2(10) = (1-\rho)/2, \quad P_2(11) = (3\rho-1)/2. \qquad (4)$$

Remember that terms of order f/p have been neglected. In the limit $f/p \to 0$, i.e., at the critical point, Eq. (4) becomes exact. Before we consider the case $n \geq 3$, we would like to comment on the results of Eq. (4).

(i) $P_2(10) = \sum_{s=1}^{\infty} n(s)$ [see (2)] since the configuration 10 represents the right-hand edge of a forest cluster.

(ii) The exact result for $f_2(10)$ is $f_2(10)P_2(10) = f\rho$ since each time lightning strikes the system a right-hand edge of a forest cluster burns down. $f_2(10)$ is a factor f/p smaller than $f_2(11)$ and therefore negligible since only a portion $\propto f/p$ of all clusters burns down before trees grow at their edge. Below we will see that the approximation $f_2(10) = 0$ leads to $f_n(1\cdots10) = 0$ for all $n > 2$, simplifying considerably the calculations. $f_n(1\cdots10) = 0$ cannot be true for large values of n since forest clusters of size p/f are struck by lightning with a large probability, which leads to $f_n(1\cdots10) \neq 0$. Our subsequent results will therefore only be valid for $n \ll p/f$.

We are now able to calculate the probabilities P_n. From $f_n(1\cdots1)P_n(1\cdots1) + f_n(1\cdots10)P_n(1\cdots10) = f_{n-1}(1\cdots1)P_{n-1}(1\cdots1)$ and $f_n(1\cdots10)P_n(1\cdots10) \leq f_{n-1}(1\cdots10)P_{n-1}(1\cdots10)$ we obtain by recursion $f_n(1\cdots1)P_n(1\cdots1) = p(1-\rho)$ and $f_n(1\cdots10) = 0$. This means that a string of n sites is covered by a completely dense forest when it catches fire. This dense forest belongs to a large forest cluster which has a mean size $\bar{s} \gg n$ at the moment when it is struck by lightning. As long as not all sites are occupied by trees, the dynamics of our string are completely determined by tree growth. All configurations which contain the same number of trees therefore have the same probability. Let $P_n(m)$ be the probability that the string is occupied by m trees. In the steady state the $P_n(m)$ are related by the equations

$$pnP_n(0) = f_n(1\cdots1)P_n(n) = p(1-\rho),$$

$$p(n-m)P_n(m) = p(n-m+1)P_n(m-1) \text{ for } m \neq 0, n,$$

which lead to the result

$$P_n(m) = (1-\rho)/(n-m) \quad \text{for } m < n,$$
$$P_n(n) = 1-(1-\rho)\sum_{m=0}^{n-1} 1/(n-m) = 1-(1-\rho)\sum_{m=1}^{n} 1/m. \tag{5}$$

The size distribution of forest clusters is

$$n(s) = P_{s+2}(01\cdots10) = \frac{P_{s+2}(s)}{\binom{s+2}{s}} = \frac{1-\rho}{(s+1)(s+2)} \simeq (1-\rho)s^{-2}. \tag{6}$$

This is a power law with the critical exponent $\tau = 2$. Equation (6) leads to $\sum_{s=1}^{\infty} n(s) = (1-\rho)/2$, in agreement with Eq. (2). The size distribution of fires is $\propto sn(s) \propto s^{-1}$. The size distribution $n_e(s)$ of clusters of empty sites is

$$n_e(s) = P_{s+2}(10\cdots01) = \frac{P_{s+2}(2)}{\binom{s+2}{2}} = \frac{2(1-\rho)}{s(s+1)(s+2)}, \tag{7}$$

which is also derived in [12].

There is a characteristic length $s_{\max}$ where the power law $n(s) \propto s^{-2}$ breaks down. During one time step, a cluster of size s grows with probability $2p$ to a bigger cluster, and it is destroyed by lightning with probability fs. When the cluster size approaches $s \propto p/f$, the cluster is struck by lightning with a finite probability fs. This simple argument leads to $s_{\max} \propto p/f$ and $\lambda = 1$, as did the scaling theory in [7]. The remaining critical exponents are $\mu = \nu = 1$, as already derived in [7]. The following microscopic derivation shows that the relation for $s_{\max}$ acquires a logarithmic correction factor. We calculate $s_{\max}$ from the condition that a string of size $n \le s_{\max}$ is not struck by lightning until all trees are grown. When a string of size n is completely empty at time $t = 0$, it will be occupied by n trees after $T(n) = (1/p)\sum_{m=1}^{n} 1/m \simeq \ln(n)/p$ time steps on an average. The mean number of trees after t time steps is $m(t) = n[1-\exp(-pt)]$. The probability that lightning strikes a string of size n before all trees are grown is $f\sum_{t=1}^{T(n)} m(t) \simeq (f/p)n[\ln(n)-1] \simeq (f/p)n\ln(n)$. We conclude

$$s_{\max}\ln(s_{\max}) \propto p/f \text{ for large } p/f. \tag{8}$$

Next we calculate the relation between the mean forest density ρ and the parameter f/p. The probability $P_n(n)$ in Eq. (5) cannot be less than zero. Since $\sum_{m=1}^{n} 1/m \simeq \ln(n)$ for large values of n, the mean forest density must approach the value 1 at the critical point via $1-\rho \propto 1/\ln(s_{\max})$. A more precise result for ρ is obtained from

$$\rho = \sum_{s=1}^{\infty} sn(s) = (1-\rho)\sum_{s=1}^{s_{\max}} \frac{s}{(s+1)(s+2)} + \sum_{s=s_{\max}+1}^{\infty} sn(s) \simeq (1-\rho)\ln(s_{\max}) + \int_{s_{\max}}^{\infty} sn(s)ds.$$

With the scaling ansatz $n(s) = (1-\rho)s^{-2}\mathcal{C}(s/s_{\max})$, the second term in the last line reduces to $1-\rho$ multiplied by a constant factor, and thus

$$\frac{\rho}{1-\rho} = \ln(s_{\max}) + \text{const} \simeq \ln(p/f) + \text{const} \simeq \ln(p/f) \quad \text{for large } p/f. \tag{9}$$

The forest density approaches the value 1 at the critical point. This is not surprising since there exists no infinitely large cluster in a one-dimensional system as long as the forest is not completely dense. Combining Eqs. (6) and (9), we obtain the final result for the cluster-size distribution near the critical point

$$n(s) \simeq \frac{1}{(s+1)(s+2)\ln s_{\max}} \text{ for } s < s_{\max}, \tag{10}$$

with $s_{\max}$ given by Eq. (8).

Our computer simulations confirm the analytic results. The simulations were performed using the same method as in [8]. In Fig. 1 the size distribution $sn(s)$ of clusters struck by lightning is plotted. The smooth line is the analytic result derived from Eq. (6). It fits the simulation perfectly in the region $s < s_{\max} \simeq 1000$. The bump in the cluster-size distribution has the following explanation: consider the dynamics of a forest cluster. As long as the cluster size is $s < s_{\max}$, its average size increases each time step by $2p(s+2)$ [can be derived from Eq. (5)]; i.e., its growth speed is $\dot{s} \propto s$. A tree spends a portion $\propto 1/\dot{s}$ of its lifetime in clusters of size s, and consequently the growth speed is related to the cluster-size distribution by $sn(s) \propto 1/\dot{s}$, which leads again to the power law $n(s) \propto s^{-2}$. When the cluster size becomes comparable to $s_{\max}$, the growth speed increases no longer proportionally to s but slower than s since the largest neighbors of the cluster have already been destroyed by lightning.

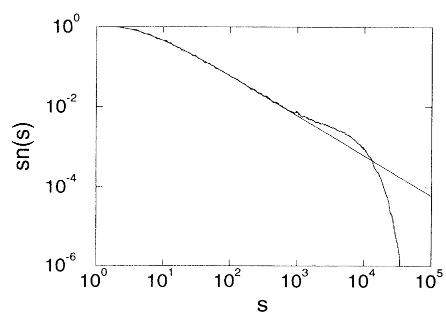

FIG. 1. Size distribution of the fires for $f/p = 2/50\,000$ and $L = 2^{20}$. The smooth line is the theoretical result which is valid for cluster sizes $\le s_{\max}$.

TABLE I. Numerical results for the ratios $\ln(p/f)(1-\rho)/\rho$ and $(p/f)/s_{\max}\ln s_{\max}$ for different values of p/f.

p/f	3125	6250	12 500	25 000	50 000
$\ln(p/f)(1-\rho)/\rho$	1.13	1.10	1.06	1.04	1.02
$(p/f)/s_{\max}\ln s_{\max}$	3.65	3.61	3.56	3.62	3.55

Therefore $s^2n(s)$ increases with increasing s for $s > s_{\max}$, giving rise to the bump in the cluster-size distribution. When the forest cluster is so large that it is struck by lightning with a nonvanishing probability, $n(s)$ decreases exponentially.

The ratio $\ln(p/f)(1-\rho)/\rho$ is given in Table I for different values of p/f. It approaches the value 1 with increasing p/f, as predicted by theory [see Eq. (9); the deviations from the value 1 for smaller p/f are due to the constant contribution in Eq. (9)]. The ratio $(p/f)/s_{\max}\ln(s_{\max})$ is also shown in Table I for different values of p/f. Within numerical accuracy these ratios are identical, in agreement with our theoretical result Eq. (8).

Finally, we calculate the temporal correlation function of the number of burning sites. The mean number $N_s(t)$ of trees that burn t time steps after a cluster of size s is struck by lightning is $N_s(t) \simeq 2(1-t/s)\theta(s-t)$. The correlation function is then

$$G(\tau) \propto \int_1^\infty ds\, n(s) s \int_0^\infty dt\, N_s(t) N_s(t+\tau) \propto 2s_{\max}/\ln(s_{\max}) - 3\tau[1-\ln(\tau)/\ln(s_{\max})]. \quad (11)$$

The Fourier transform of $G(\tau)$ is $G(\omega) \propto \omega^{-2}[1+\text{const}\times\ln(\omega s_{\max})]$ for small $\omega > 1/s_{\max}$. There is a nontrivial deviation from the ω^{-2} dependence in the direction of $1/f$ noise. Our simulations confirm the power law dependence (Fig. 2) but cannot discriminate between a ω^{-2} law and the complete expression Eq. (11).

We conclude with three comments.

(i) In the limit $f/p \to 0$ only an infinitesimal portion of all forest clusters, all empty sites, and even all trees are not described by our results Eqs. (5)–(7) which therefore become exact at the critical point.

(ii) The self-organized critical forest-fire model is in one dimension much simpler than in higher dimensions. The critical exponents are classical in one dimension, and the tree distribution on a string of size $n \le s_{\max}$ is stochastic. In higher dimensions, there are always trees left when a fire passes through a region. Therefore the tree distribution is not stochastic and the exponents are nontrivial.

(iii) The only conditions for obtaining the result Eq. (5) are that trees grow stochastically and that a fire does not pass through a string of size $n \le s_{\max}$ as long as not all its trees are grown. The P_n and consequently the cluster-size distributions remain unchanged when the rules for the occurrence of lightning are changed in one of the following ways: (a) Lightning strikes all forest clusters of size $\ge s_{\max}$. This model is critical in the limit $s_{\max} \to \infty$. (b) Lightning strikes every T time steps the largest forest cluster on each interval of l sites. This version is critical for $l \to \infty$ with fixed T. (c) Lightning strikes a forest cluster of size s with a probability $\epsilon f(s)$ which does not decrease with increasing s. This model is critical for $\epsilon \to 0$. While the size distribution of the forest clusters and of the clusters of empty sites remains unchanged in all these versions of the forest-fire model, the size distribution of fires varies considerably and is in general no power law.

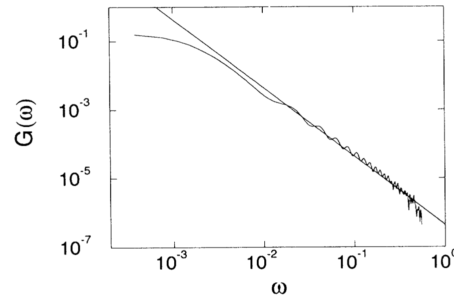

FIG. 2. The power spectrum $G(\omega)$ of the fire density. The simulation parameters are $f/p = 8/50\,000$ and $L = 2^{20}$. The straight line (theory) has the slope -2.

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