Solution of an Orbital Kondo Array

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We introduce a solvable model of a one-dimensional electron gas interacting with an array of dynamical scattering centers, whose state is specified by a pseudospin variable. In the dilute limit, for frequencies ω and temperatures T below the single-center Kondo scale but above a coherence scale Δ, the physics is governed by the fixed point of the single-impurity two-channel Kondo problem, and the physical properties are reminiscent of the normal state of the high-temperature superconductors. As ω and $T \to 0$, three susceptibilities are equally divergent: (1) conventional, spin-singlet even-parity pairing, (2) composite spin-singlet odd-parity pairing, and (3) odd-parity pseudospin.

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In this paper we consider a one-dimensional version of a model we have used previously [1] to express the consequences of frustrated phase separation in high-temperature superconductors. It is a solvable limit of a one-dimensional electron gas interacting with an array (which may or may not be ordered) of dynamical scattering centers and, as such, it is also of considerable interest in its own right. The model is an orbital analog of the Kondo problem in which the scattering center is specified by a pseudospin variable which couples to the orbital states of the electron gas rather than to their true spin. Previous studies have considered a single scattering center: Here we generalize the model *and* the solution to an array.

We shall work directly with the field-theory version of the model, although it is easy to define an equivalent lattice model [2]. The Hamiltonian is

$$H = H_0 + H_1 , \tag{1}$$

where H_0 is the kinetic energy:

$$H_0 = iv_F \sum_\sigma \int dr [\psi^\dagger_{1,\sigma} \partial_r \psi_{1,\sigma} - \psi^\dagger_{2,\sigma} \partial_r \psi_{2,\sigma}] . \tag{2}$$

Here $\psi^\dagger_{1,\sigma}(r)$ and $\psi^\dagger_{2,\sigma}(r)$ create, respectively, a right- and left-moving electron at position r and with z component of spin $\sigma = \pm \frac{1}{2}$. H_1 is the coupling between the electron gas and the dynamical scattering centers:

$$H_1 = J_\parallel a \sum_{R,\sigma} \tau^z_R [\psi^\dagger_{1,\sigma}(R)\psi_{1,\sigma}(R) - \psi^\dagger_{2,\sigma}(R)\psi_{2,\sigma}(R)] + J_\perp a \sum_{R,\sigma} [\tau^-_R e^{2ik_F R} \psi^\dagger_{2,\sigma}(R)\psi_{1,\sigma}(R) + \text{H.c.}] , \tag{3}$$

where τ^λ_R are the pseudospin-half operators representing the degrees of freedom of the scattering center at position R, a is a lattice constant, and k_F is the Fermi wave vector. We assume throughout this paper that $J_\perp \ll W$, where $W \approx \pi v_F/a$ is the bandwidth.

It is also assumed that τ has the symmetry of a local dipolar model [1]; in particular, τ^z_R couples to the local electronic current, and hence, like the current, it must be odd under time reversal and parity. This coupling is an important feature of our model, and it is responsible for some of the unusual consequences.

It is easy to see that there are two conserved quantities: the total charge of the conduction band,

$$Q = \int dr \sum_\sigma [\psi^\dagger_{1,\sigma}\psi_{1,\sigma} + \psi^\dagger_{2,\sigma}\psi_{2,\sigma}]$$

and a conserved current $Q_J = \int dr\, \rho_J(r)$, where

$$\rho_J(r) = \sum_\sigma [\psi^\dagger_{1,\sigma}\psi_{1,\sigma} - \psi^\dagger_{2,\sigma}\psi_{2,\sigma}] + \sum_R \tau^z_R \delta(r - R) . \tag{4}$$

In the limit of a single scattering center at the origin, the model may be mapped into the two-channel Kondo problem by replacing $\psi_{2,\sigma}(r)$ by the right-going field $\psi_{2,\sigma}(-r)$ [3]. Then the labels (1,2) and σ may be regarded as pseudospin and flavor, respectively, and the Hamiltonian defined in Eqs. (1)–(3) is precisely the anisotropic two-channel Kondo problem, solved in a previous paper for the "Toulouse limit" $\Delta J_\parallel \equiv J_\parallel - \pi v_F/a = 0$ [4]. All of the results obtained before [4] carry over unchanged to the present problem: In particular, all of the known *universal* properties of the isotropic ($J_\parallel = J_\perp$) two-channel Kondo problem [5] are shared by this model [6]. The low-frequency and low-temperature behavior is governed by a single nonuniversal energy scale Γ (the Kondo scale) which, in the Toulouse limit, is $\Gamma = J_\perp^2 a/\pi v_F$.

Turning now to the array, we may anticipate some aspects of the solution on general grounds. There are two important energy scales: the single-center Kondo scale Γ and a coherence scale Δ, which characterizes the induced interactions between neighboring pseudospins. Δ is a calculable function of $J_\perp/v_F$ and the concentration of pseudospins, c. Since $\Delta \to 0$ as $c \to 0$, there must be a dilute limit $c \ll c_1$ in which $\Gamma \gg \Delta$. In that limit, there are three distinct regimes of temperature: (1) $T > \Gamma$, where the scattering from the pseudospins is weak; (2) $\Delta \ll T \ll \Gamma$, where *universal* single-impurity Kondo physics is observed; and (3) $T \ll \Delta$, where the pseudospins form a state with long-range coherence, in which the critical

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fluctuations associated with a single impurity are cut off and, as we shall see, long-range pairing fluctuations are enhanced. The opposite extreme is the dense limit $c \gg c_2$, where $\Delta \gg \Gamma$ and hence, at $T \sim \Delta$, the system crosses directly from the high-temperature disordered state to the low-temperature pseudospin-coherent state. We will show below that both c_1 and c_2 are $\sim \Gamma/W$; hence, there need not be a large crossover region between the dilute and dense limits.

We now show how this picture may be derived by using a combination of exact results and well-controlled approximations. The trick of replacing $\psi_{2,\sigma}(r)$ by a right-going field does not work for the array: Consequently, we proceed by bosonizing the Hamiltonian in its orignal form, using the standard one-dimensional relation between Bose and Fermi fields [7],

$$\psi^{\dagger}_{a,\sigma}(r) = [2\pi a]^{-1/2} \exp\{i\Phi_{a,\sigma}(r)\}, \tag{5}$$

where $\Phi_{b,\sigma} = \sqrt{\pi}[\int^{r} dr' \Pi_{\sigma}(r') \pm \phi_{\sigma}(r)]$ (with lower and upper signs corresponding to $b = 1$ and 2, respectively) and $\phi_{\sigma}(x)$ and $\Pi_{\sigma}(x)$ are canonically conjugate Bose fields satisfying $[\phi_{\sigma}(r), \Pi_{\sigma'}(r')] = i\delta(r - r')\delta_{\sigma,\sigma'}$. Then the Hamiltonian may be written in terms of a spin field $\phi_s(r) = [\phi_{\uparrow} - \phi_{\downarrow}]/\sqrt{2}$ and a charge field $\phi_c(r) = [\phi_{\uparrow} + \phi_{\downarrow}]/\sqrt{2}$, together with their conjugate momenta Π_s and Π_c, in the form $H = H_c(J_{\parallel}) + H_s[\theta_R]$ where

$$H_c(J_{\parallel}) = \frac{v_F}{2} \int dr [\Pi_c^2 + (\phi_c')^2] + J_{\parallel} a \sqrt{2/\pi} \sum_R \tau^z_R \Pi_c(R), \tag{6}$$

with $\phi' \equiv \partial_r \phi$, and

$$H_s[\theta_R] = \int dr \left\{ \frac{v_F}{2} [\Pi_s^2 + (\phi_s')^2] - \mathcal{J}_{\perp}(r) \cos(\beta \phi_s) \right\}. \tag{7}$$

Here

$$\mathcal{J}_{\perp}(r) = (-2J_{\perp}/\pi) \sum_R \hat{\mathbf{e}}(\theta_R) \cdot \boldsymbol{\tau}_R \delta(r - R)$$

and

$$\hat{\mathbf{e}}(\theta_R) = \hat{\mathbf{e}}_x \cos(\theta_R) + \hat{\mathbf{e}}_y \sin(\theta_R),$$

with $\theta_R = 2k_F R + \sqrt{2\pi}\phi_c(R)$. It will be useful to consider arbitrary values of β in $H_s[\theta_R]$, although $\beta^2 = 2\pi$ for the model as defined. The Hamiltonian H_c is independent of the spin fields, and H_s depends on the charge fields only through the appearance of $\phi_c(R)$ in θ_R. Consequently, it is possible to separate spin and charge by making a unitary transformation to rotate the τ spins through an angle $[-\sqrt{2\pi}\phi_c(R)]$ about the $\hat{\mathbf{e}}_z$ direction:

$$U^{\dagger} H U = H_c(\Delta J_{\parallel}) + H_s[\theta^0_R] + \text{const}, \tag{8}$$

where $U = \exp[-i\sqrt{2\pi}\sum_R \phi_c(R)\tau^z_R]$ and $\Delta J_{\parallel} = J_{\parallel} - \pi v_F/a$. The transformed Hamiltonian involves $H_s[\theta^0_R]$, where $\theta^0_R \equiv 2k_F R$ is independent of the charge fields.

When $\Delta J_{\parallel} = 0$, the canonical transformation U removes the τ pseudospins entirely from the charge part of the Hamiltonian H_c in Eq. (6) and from the conserved current ρ_J in Eq. (4). Moreover, the operators $\hat{\mathbf{e}}(\theta^0_R) \cdot \boldsymbol{\tau}_R$ commute with the transformed Hamiltonian so their eigenvalues may be used to label the eigenstates. Thus coherence of the array is achieved via an Ising ordering of the pseudospins. In fact, it will be seen that the effective interaction between pseudospins favors the "uniform state" in which $\hat{\mathbf{e}}(\theta^0_R) \cdot \boldsymbol{\tau}_R = \frac{1}{2}$ [or equivalently $\hat{\mathbf{e}}(\theta^0_R) \cdot \boldsymbol{\tau}_R = -\frac{1}{2}$] for all R. It is important to stress that this result does *not* mean that the physical pseudospins are ordered. The point is that $\hat{\mathbf{e}}(\theta^0_R) \cdot \boldsymbol{\tau}_R = U^{\dagger} \hat{\mathbf{e}}(\theta_R) \cdot \boldsymbol{\tau}_R U$: Thus in terms of the *untransformed* variables, the uniform state has $\hat{\mathbf{e}}(\theta_R) \cdot \boldsymbol{\tau}_R = \frac{1}{2}$ for all R. But $\hat{\mathbf{e}}(\theta_R) \cdot \boldsymbol{\tau}_R$ depends on both $\boldsymbol{\tau}_R$ and $\phi_c(R)$; i.e., it is a *composite* operator, involving the physical pseudospins and the charge density of the conduction electrons. Moreover, this composite ordering is not easily observable because it does not have a simple expression in terms of the original *fermion* variables. From these results it also follows that, at low enough temperatures,

$$\langle \boldsymbol{\tau}_R \cdot \boldsymbol{\tau}_{R'} \rangle_U \sim \cos[2k_F(R - R')] e^{-|R - R'|/\xi_{\tau}(T)}, \tag{9}$$

where $\langle \rangle_U$ is the thermal expectation value with respect to the transformed Hamiltonian $U^{\dagger} H U$, and $\xi_{\tau}(T) \sim e^{\Delta/T}$. Here Δ, whose value will be estimated below, is the energy to create a kink in the pseudospin ordering such that $\hat{\mathbf{e}}(\theta^0_R) \cdot \boldsymbol{\tau}_R = -\frac{1}{2}$ for $R < 0$ and $\hat{\mathbf{e}}(\theta^0_R) \cdot \boldsymbol{\tau}_R = +\frac{1}{2}$ for $R \geq 0$.

A number of general properties of the model may be inferred by using the idea that the low-energy, long-wavelength properties are governed by a single zero-temperature fixed point, or a few possible isolated fixed points. Any solvable model in the basin of attraction of the relevant fixed point will have the same universal low-energy behavior as the system of physical interest. For the orbital Kondo array, after the canonical transformation (8), the spin fields are described by a generalized sine-Gordon Hamiltonian $H_s[\theta^0_R]$ in which the cosine term is finite at a discrete set of points R. As for the ordinary sine-Gordon model, the Hamiltonian may be mapped into free fermions at the point $\beta^2 = 4\pi$, and reduces to free bosons for $\beta \to 0$ [7]. The problem may be solved exactly in these limits, and it is found that both have the same low-energy physics, apart from the energy scale. Thus, the same fixed point must also govern the low-energy physics for the case of physical interest, $\beta^2 = 2\pi$, unless it is separated from *both* the free-fermion point *and* the free-boson region by phase transitions.

It follows that the physical properties of the model defined in Eqs. (1)–(3) may be deduced by solving either the free-fermion or free-boson model and then using scaling [7] or renormalization-group [8] arguments to obtain the energy scale as a function of β. In this way it is found that the model possesses the following properties: (1) At $T = 0$, the low-lying excitations (solitons) are massive fermions [7] with energy spectrum $E_s(k) = \pm[(v_F k)^2 + \Delta_s^2]^{1/2}$, where Δ_s will be evaluated below for dilute and

dense impurity concentrations. (2) There is a corresponding correlation length, $\xi_s = v_F/\Delta_s$, which determines the length scale over which the ground state can be deformed. (3) The energy scale of the pseudospin kink is finite so at zero temperature the transformed pseudospins are ordered.

It is possible to obtain further quantitative results in various limits. Of these, the easiest to treat is the high-density limit in which $c\xi_s(c,J_\perp)/a \gg 1$. Here the "granular" character of $\mathcal{J}_\perp(r)$ may be ignored so that, for the ground state configuration of the pseudospins, we may approximate $\mathcal{J}_\perp(r) \approx J_\perp c$, while for the one-kink configuration, $\mathcal{J}_\perp(r) \approx J_\perp c\Theta(r)$. Once this approximation is made, H_s is identical to the quantum sine-Gordon Hamiltonian and we may use known exact results [7,8] to show that $\Delta_s = \Delta_s^0[1+\mathcal{O}((J_\perp/W)^2)]$, where

$$\Delta_s^0 = Wf(\beta)[cJ_\perp/W]^{4\pi/[8\pi-\beta^2]}, \tag{10}$$

and $f(\beta)$ is a nonuniversal function of $\mathcal{O}(1)$. Similarly, the pseudospin interaction energy $\Delta \sim \Delta_s$. The regime of validity of the dense limit may be determined self-consistently by solving the implicit equation $c_1\xi_s(c_1,J_\perp)/a \sim 1$ to find

$$c_1 \sim [J_\perp/W]^{4\pi/[4\pi-\beta^2]}. \tag{11}$$

In particular, for $\beta^2 = 2\pi$, $c_1 \sim [J_\perp/W]^2 \sim \Gamma/W$, and $\Delta_s(c_1) \approx \Gamma$ [4]. Thus, the dense limit may also be regarded as the range of concentrations for which $\Delta_s > \Gamma$.

In the dilute limit, $c \ll c_2$, the granular character of $\mathcal{J}_\perp$ is essential. The energy of a single pseudospin kink, which is equal to the energy of a compensating distortion of ϕ_s, depends only on bulk properties, since the distance between neighboring scattering centers is large compared to any intrinsic length scale in the problem. Specifically, the energy $\Delta(|R-R'|)$ to create a pseudospin kink between the scattering centers at sites R and R' may be evaluated in powers of $1/|R-R'|$ to give

$$\Delta(|R-R'|) = \frac{\pi v_F}{4|R-R'|}\{1+\mathcal{O}(v_F/J_\perp|R-R'|)\}. \tag{12}$$

The characteristic energy of an isolated scattering center is the Kondo scale Γ. Therefore the dilute limit requires that $\Delta(b) \ll \Gamma$, where $b \sim a/c$ is the average spacing between scattering centers. Using Eq. (12), this condition is equivalent to $c < c_2$, where $c_2 \sim \Gamma/W \sim [J_\perp/W]^2$. Note that, as stated above, there is a single crossover concentration $c_1 \sim c_2$ for the physically interesting case $\beta^2 = 2\pi$. We will show elsewhere [2] that $\Delta_s \sim \Delta$ in both the dilute and dense limits.

The low-temperature ($T < \Delta$) behavior is governed by the fluctuations of the various order parameter fields. A one-dimensional system does not have a broken continuous symmetry but may have a zero-temperature critical point at which the space-time correlation functions $C_\alpha(r,r',t-t') = \langle O_\alpha(r,t)O_\alpha(r',t')\rangle$ of operators $O_\alpha(r,t)$ are power laws, indicating which type of long-range order might be realized in higher dimensions. (The system is not invariant under a space translation so the correlation functions will, in general, depend on r and r' separately.) In the Toulouse limit, three distinct operators have correlation functions which diverge in the same way when $T \to 0$: (1) conventional singlet, even-parity superconducting pairing, $O_{SS} = [\psi^\dagger_{1,\uparrow}\psi^\dagger_{2,\downarrow} + \psi^\dagger_{2,\uparrow}\psi^\dagger_{1,\downarrow}]$; (2) composite, spin-singlet, odd-parity superconducting pairing, $O_{\text{comp}}(r,t) = \psi^\dagger_{2,\downarrow}(r,t)\psi^\dagger_{2,\uparrow}(r,t)\tau^+_{R_r}(t)$, where R_r signifies the pseudospin site closest to position r; and (3) the orientation of the pseudospins, $\mathbf{O}_\tau = \boldsymbol{\tau}_R$. These results may be derived by introducing the boson representation (5) and carrying out the canonical transformation (8) to show that the dominant long-time, long-distance contributions to the correlation functions are proportional to the product of $\langle \boldsymbol{\tau}_R \cdot \boldsymbol{\tau}_{R'}\rangle_U$ and $G_{j,c} = \langle \psi^\dagger_{j,c}(r,t) \times \psi_{j,c}(0,0)\rangle_U$. The former is defined in Eq. (9); it falls exponentially over a length scale ξ_τ which itself diverges exponentially with decreasing temperature. Consequently, when $T < \Delta$, the dominant decay of all three order parameter correlation functions is determined by G_{jc}; all fall as $1/|r|$ and their Fourier transforms diverge like $1/T$ as $T \to 0$.

In order to evaluate the electrical conductivity, we adopt the most natural definition of the current operator in the context of the one-dimensional model. First of all, we define a "free charge" as $e\sum_\sigma[\psi^\dagger_{1,\sigma}\psi_{1,\sigma} + \psi^\dagger_{2,\sigma}\psi_{2,\sigma}]$, and a "bound charge" associated with the pseudospins: $\rho_{\text{bound}}(r) = e^*\sum_R \tau^x_R \partial_r \delta(r-R)$ (i.e., we treat the pseudospins as electric dipoles). The free-charge current may be deduced from the equation of continuity,

$$j(r) = v_F\rho_J + \frac{2}{\pi}\Delta J_\parallel a\sum_R \tau^z_R\delta(r-R), \tag{13}$$

where ρ_J is defined in Eq. (4). This has the property that, for $\Delta J_\parallel = 0$, the current is conserved and hence the critical amplitude of the interesting contribution to the conductivity vanishes. In order to obtain the generic behavior it is necessary to perturb about the Toulouse limit. The Kubo formula relates the conductivity to $1/\omega$ times the Fourier transform of the current susceptibility:

$$C(r,r',t) \equiv i\Theta(t)\langle[j(r,t),j(r')]\rangle. \tag{14}$$

After the canonical transformation (8), the two parts of $j(r)$ in Eq. (13) are uncorrelated, so they give separate contributions to $C(r,r',t)$. Since ρ_J is a conserved quantity, it leads to a $\delta(\omega)$ term in the frequency-dependent conductivity. This delta function is replaced by a broad "Drude" peak $\sigma_D(\omega)$ in the presence of static disorder and/or electron-phonon coupling, but it is unaffected by the dynamics of the pseudospins, from which it is decoupled. The second term in Eq. (13) gives an anomalous contribution to the conductivity:

$$\sigma_a(\omega,T) = ie^2\left[\frac{2a\Delta J_\parallel}{\pi}\right]^2\frac{\chi_{z,z}(\omega,T)}{\omega}, \tag{15}$$

where

$$\chi_{a,b}(\omega,T)=\frac{1}{L}\sum_{R,R'}\int_0^\infty dt\, e^{i\omega t}\langle[\tau_R^a(t),\tau_{R'}^b]\rangle\,, \tag{16}$$

with L equal to the length of the system. Similarly we find that the bound-charge contribution to the conductivity is given by

$$\sigma_b(\omega,T)=i(e^*)^2\omega\chi_{x,x}(\omega,T)\,. \tag{17}$$

Since $\sigma_a(\omega,T)$ and $\sigma_b(\omega,T)$ depend only on the dynamics of the pseudospins, they are insensitive to other perturbations to the model, such as electron-phonon scattering.

In the dilute incoherent limit ($\Delta\ll T$ and $c\ll c_1$), the different pseudospins are essentially independent, so we may use the general single-impurity results obtained previously [4,9,10],

$$\chi''_{a,b}(\omega)\sim c\delta_{a,b}\tanh(\beta\omega/2)\frac{\Gamma}{\Gamma^2+\omega^2}\,, \tag{18}$$

valid for arbitrary $J_\parallel$ and for $\omega,T\ll W$. From Eqs. (15) and (18) it may be seen that $\sigma_a(\omega,T)\sim 1/T$ if $\omega\ll T\Gamma$ and $\sim 1/\omega$ if $T\ll\omega\ll\Gamma$. Moreover, $\sigma_a(\omega,T)$ is *directly* proportional to the impurity concentration c, and thus it is best regarded as a *paraconductivity* reflecting enhanced composite pairing correlations over a wide range of temperature and frequency [4]. (Note these expressions for the conductivity are qualitatively different from those obtained by Ludwig and Affleck [5], as they should be since they relate to different realizations of the two-channel Kondo problem with different forms of the current operator.) It is more difficult to obtain a simple closed form expression for the conductivity when $T<\Delta$. A mean-field estimate [2] for $T=0$, where the *transformed* pseudospins are ordered and experience a Weiss field h, gives

$$\sigma_a\sim c\Gamma\omega/[h^4+\omega^2(\Gamma^2-h^2)+\omega^4]\,.$$

Consequently, $\sigma_a\sim c\omega/h^4$, when $\omega\ll h^2/\Gamma$.

Finally, we comment on the nature of the array of scattering centers in the case where they are annealed. For strong repulsive interactions between scatterers, we expect a roughly uniformly spaced lattice. However, we find that there is an induced interaction between scattering centers that favors dimerization or even phase separation under appropriate circumstances. The coherent low-temperature physics is the same in the dilute and dense limits, and is insensitive to the precise distribution of the scattering centers; the distribution is important only at intermediate temperatures.

The finite-temperature properties of the dilute orbital Kondo lattice are remarkably reminiscent of the observed normal state properties of the high-temperature superconductors [1]. In particular, we have found a T-linear and ω-linear contribution to the resistivity which is insensitive to all other details, including scattering from ordinary impurities and phonons. Moreover, we have found a crossover at a temperature $T\sim\Delta$ to a low-temperature state in which the pseudospins are coherent and correlations of two distinct pairing operators are substantially enhanced. In higher dimensions, where the crossover becomes a true phase transition, the physics of the orbital Kondo lattice might lead to different kinds of superconducting order parameter, depending on the details of the model and possibly on additional interactions not included in the Hamiltonian studied here. In a forthcoming publication we shall expand on these results and further discuss their relevance for high-temperature superconductors.

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