Bose-Fermi Transformation in Three-Dimensional Space

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A generalization of the Jordan-Wigner transformation to three (or higher) dimensions is constructed. The nonlocal mapping of spin to fermionic variables is expressed as a gauge transformation with topological charge equal to 1. The resulting fermionic theory is minimally coupled to a non-Abelian gauge field in a spontaneously broken phase containing monopoles.

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The Jordan-Wigner (JW) transformation [1] for one-dimensional spin systems has provided remarkable applications in condensed matter physics, including the two-dimensional classical Ising model [2,3] and the XY spin-1/2 model [4]. The counterpart in relativistic field theory, the bosonization of fermionic theories in 1+1 dimensions [5], has also opened an important field of active research.

Bosonization in higher dimensions has been elusive for a long time. Relatively recent work has uncovered a Bose-Fermi transmutation in 2+1 dimensions which is experienced by the elementary excitations of the sigma model in the presence of a Chern-Simons field [6,7]. This result paved the way for the construction of the JW transformation in a lattice of two spatial dimensions, where a local fermion theory is mapped onto a system of hard-core bosons described by the Heisenberg Hamiltonian [8]. On the same basis, the bosonization scheme has been also implemented for 2+1 relativistic field theory [9].

In this Letter, we propose an extension of the JW transformation to three—or more—dimensions. Here we discuss in detail the three-dimensional case. The generalization to higher dimensions is straightforward.

The JW transformation relates the local spin-1/2 operators, $S^z, S^\pm$ ($[S^z,S^\pm]=\pm S^\pm$, $[S^+,S^-]=2S^z$), to local fermionic operators, ψ, $\psi^\dagger$ ($\{\psi,\psi^\dagger\}=1$, $\{\psi,\psi\}=\{\psi^\dagger,\psi^\dagger\}=0$):

$$S^+(\mathbf{x})=\psi^\dagger(\mathbf{x})\,U(\mathbf{x})\,,\quad S^-(\mathbf{x})=U^\dagger(\mathbf{x})\psi(\mathbf{x})\,,\qquad (1)$$

where $U(\mathbf{x})$ is a nonlocal function of ψ.

In a one-dimensional lattice (1D), the operator U takes the form

$$U_{1\mathrm{D}}(\mathbf{x})=e^{i\pi\sum_{\mathbf{z}\neq\mathbf{x}}j^0(\mathbf{z})\theta(\mathbf{z}-\mathbf{x})}\,,\qquad (2)$$

where $j^0(\mathbf{z})=\psi^\dagger(\mathbf{z})\psi(\mathbf{z})$ is the fermion number density operator, and $\theta(\mathbf{z})$ is the 1D step function. The corresponding expression in 2D is [8,10]

$$U_{2\mathrm{D}}(\mathbf{x})=e^{i\sum_{\mathbf{z}\neq\mathbf{x}}j^0(\mathbf{z})\arg(\mathbf{z}-\mathbf{x};\hat{\mathbf{n}}_0)}\,,\qquad (3)$$

where the function $\arg(\mathbf{x})$ is the angle between $\mathbf{x}$ and an arbitrarily given space direction $\hat{\mathbf{n}}_0$.

The 3D JW transformation has the same form as (1), but now it connects an SU(2) doublet of spins S_α to an SU(2) doublet of fermion operators ψ_α:

$$S^+_\alpha(\mathbf{x})=\psi^\dagger_\alpha(\mathbf{x})e^{i\sum_{\mathbf{z}\neq\mathbf{x}}j^{0a}(\mathbf{z})\omega^a(\mathbf{z}-\mathbf{x};\hat{\mathbf{n}}_0)}\,,\qquad (4)$$

where $j^{0a}(\mathbf{z})\equiv\psi^\dagger(\mathbf{z})\tau^a\psi(\mathbf{z})$ is an SU(2) "isospin" density operator ($[\tau^a,\tau^b]=i\epsilon^{abc}\tau^c$, sum over repeated indices is implied), and

$$\omega^a(\mathbf{x};\hat{\mathbf{n}}_0)=\arg(\mathbf{x};\hat{\mathbf{n}}_0)\,e^a(\mathbf{x};\hat{\mathbf{n}}_0)\,,\qquad (5)$$

with $e^a(\mathbf{x};\hat{\mathbf{n}}_0)$ being a unit vector orthogonal to $\mathbf{x}$ and $\hat{\mathbf{n}}_0$. The application $\mathbf{x}\to\omega^a(\mathbf{x};\hat{\mathbf{n}}_0)$ generalizes the 2D and 1D expressions, as can be seen by restricting it to a plane and to a line, respectively.

The mapping is completed by exhausting the commutator algebra of S^+_α and S^-_α. In particular, the generalization of S^z, $S^z_{\alpha\beta}(\mathbf{x})\equiv(1/2)[S^+_\alpha(\mathbf{x}),S^-_\beta(\mathbf{x})]$, is

$$S^z_{\alpha\beta}=\frac{1}{2}[1-\rho(\mathbf{x})]\delta_{\alpha\beta}-\frac{1}{2}j^{0a}(\mathbf{x})\tau^a_{\alpha\beta}\,,\qquad (6)$$

where $\rho(\mathbf{x})\equiv\psi^\dagger_\alpha(\mathbf{x})\psi_\alpha(\mathbf{x})$ is the fermion density. It is readily seen that the diagonal part, $S^z_{\alpha\alpha}$, has the usual form, $1/2-\psi^\dagger_\alpha\psi_\alpha$ (no sum over α). The inverse of (4) reads

$$\psi^\dagger_\alpha(\mathbf{x})=S^+_\alpha(\mathbf{x})\,e^{-i\sum_{\mathbf{z}\neq\mathbf{x}}S^{za}(\mathbf{z})\omega^a(\mathbf{z}-\mathbf{x};\hat{\mathbf{n}}_0)}\,,\qquad (7)$$

with $S^{za}\equiv-S^z_{\alpha\beta}\tau^a_{\beta\alpha}$.

The key feature of the ansatz (4) and (5), which is responsible for the transmutation of statistics, is the fact that

$$\omega^a(\mathbf{y}-\mathbf{x};\hat{\mathbf{n}}_0)-\omega^a(\mathbf{x}-\mathbf{y};\hat{\mathbf{n}}_0)=\pi\,e^a(\mathbf{x}-\mathbf{y};\hat{\mathbf{n}}_0)\,.\qquad (8)$$

This gives rise to a (-1) factor when the positions of two spins are exchanged, leading to opposite statistics for the S and ψ operators. In contrast with the 2D case, there are no intermediate possibilities between Bose and Fermi statistics because the structure constants of the group completely fix the normalization of the generators. For $\mathbf{x}\neq\mathbf{y}$, and using that $\{\psi_\alpha(\mathbf{x}),\psi^\dagger_\beta(\mathbf{y})\}=\delta_{\alpha\beta}\delta(\mathbf{x},\mathbf{y})$,

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$\{\psi_\alpha(\mathbf{x}),\psi_\beta(\mathbf{y})\} = \{\psi^\dagger_\alpha(\mathbf{x}),\psi^\dagger_\beta(\mathbf{y})\} = 0$, $[j^{0a}(\mathbf{x}),\psi_\alpha(\mathbf{y})] = -\tau^a_{\alpha\beta}\psi_\beta(\mathbf{x})\delta(\mathbf{x},\mathbf{y})$, $[j^{0a}(\mathbf{x}),\psi^\dagger_\alpha(\mathbf{y})] = \tau^a_{\beta\alpha}\psi^\dagger_\beta(\mathbf{x})\delta(\mathbf{x},\mathbf{y})$, one finds

$$\begin{aligned}S^-_\alpha(\mathbf{x})S^+_\rho(\mathbf{y})[e^{i\tau^a\omega^a(\mathbf{y}-\mathbf{x};\hat{\mathbf{n}}_0)}]_{\rho\beta} - {}& S^+_\beta(\mathbf{y})S^-_\rho(\mathbf{x})[e^{i\tau^a\omega^a(\mathbf{y}-\mathbf{x};\hat{\mathbf{n}}_0)}]_{\alpha\rho} \\ &= \psi_\alpha(\mathbf{x})\psi^\dagger_\beta(\mathbf{y})U^\dagger(\mathbf{x})U(\mathbf{y}) - \psi^\dagger_\beta(\mathbf{y})\psi_\rho(\mathbf{x})[e^{i\tau^a\omega^a(\mathbf{y}-\mathbf{x};\hat{\mathbf{n}}_0)}e^{-i\tau^a\omega^a(\mathbf{x}-\mathbf{y};\hat{\mathbf{n}}_0)}]_{\alpha\rho}U(\mathbf{y})U^\dagger(\mathbf{x})\,. \end{aligned} \quad (9)$$

By virtue of (8), the exponential on the right hand side (RHS) of (9) is $-\delta_{\alpha\rho}$. On the other hand, one may choose the reference vector $\hat{\mathbf{n}}_0 = (\mathbf{y}-\mathbf{x})/|\mathbf{y}-\mathbf{x}|$, making the exponentials on the LHS of (9) equal to the identity. We assume the Hamiltonian under consideration to be invariant under global SU(2) rotations. Since $\tau^a_{\alpha\beta}$ is invariant under a combination of a rotation in the "space" index a and of the isospin indices $(\alpha\beta)$, a rotation of $\hat{\mathbf{n}}_0$ can always be compensated with a global SU(2) transformation. Also, $U^\dagger(\mathbf{x})$ and $U(\mathbf{y})$ commute because the vectors $\mathbf{z}-\mathbf{y}$, $\mathbf{z}-\mathbf{x}$, and $\hat{\mathbf{n}}_0$, for a generic point $\mathbf{z}$, lie on the same plane. Hence, for different sites $\mathbf{x}$ and $\mathbf{y}$,

$$[S^-_\alpha(\mathbf{x}),S^+_\rho(\mathbf{y})] = \{\psi_\alpha(\mathbf{x}),\psi^\dagger_\beta(\mathbf{y})\}U^\dagger(\mathbf{x})U(\mathbf{y}) = 0\,. \quad (10)$$

The rest of the $\mathbf{x}\neq\mathbf{y}$ commutators can be shown to vanish along similar lines. On the other hand, the equal-site commutators define the algebra of the spin operators, which is a generalized spin-1/2 algebraic structure [11].

The essential feature of the mapping, responsible for the statistical transmutation, is its topological structure. The operators U in (2) and (3) produce local phase transformations for the field $\psi(\mathbf{x})$, generated by the charge density j^0. The U's rotate the phase of ψ in a prescribed manner at each site, throughout the entire lattice. In 1D, the resulting configuration is a kink centered at $\mathbf{x}$, where the ψ fields on the left of $\mathbf{x}$ are flipped with respect to those on the right. The 2D operator, on the other hand, produces a vortex centered at $\mathbf{x}$.

These local assignments are operations generated by j^0 in the corresponding internal symmetry groups of the fermions [$\mathbf{Z}_2$ and U(1), respectively]. Although these are gauge transformations, they cannot be continuously deformed to the identity due to their nontrivial homotopical character. The JW transformation belongs to the homotopy class of winding number one of the gauge group [12,13].

Indeed, the JW transformation establishes a one-to-one correspondence between the points of the boundary of the lattice (spatial infinity) and the elements of a group manifold. In 1D, the boundary $\{-\infty,+\infty\}$ is mapped onto $\mathbf{Z}_2$; for 2D, the circle at infinity, S^1_∞, is mapped onto U(1). The existence of these mappings not continuously connected to the identity is guaranteed because the zeroth and first homotopy groups of $\mathbf{Z}_2$ and U(1) [$\pi_0(Z_2)$ and $\pi_1(\mathrm{U}(1))$, respectively] are nontrivial.

The generalization of this construction to 3D, then, calls for a mapping between the boundary of three-dimensional space—the sphere at infinity S^2_∞—and a group manifold $\mathcal{M}$ with a nontrivial second homotopy group, $[\pi_2(\mathcal{M})\neq 0]$. The simplest choice is $\mathcal{M} = S^2 \cong \mathrm{SU}(2)/\mathrm{U}(1) \cong \mathrm{SO}(3)/\mathrm{SO}(2)$, and one is naturally led to consider the SU(2) or SO(3) gauge symmetry groups, in a spontaneously broken phase. Unlike the 1D and 2D cases, in 3D one is forced to consider a manifold that is not a Lie group. This is so because $\pi_2(G)\equiv 0$ for any Lie group (see, e.g., [13]). For higher dimensions, the requirement is $\pi_{D-1}(\mathcal{M})\neq 0$, and $\mathcal{M} = \mathrm{SO}(D)/\mathrm{SO}(D-1)$, leads one to look for spontaneously broken SO(D) gauge symmetry.

In sum, $U(\mathbf{x})$ in the ansatz (4) and (5) is a gauge transformation in the homotopy class of winding number one that, acting on a uniform configuration, produces a "hedgehog" arrangement centered at $\mathbf{x}$.

The 3D Jordan-Wigner transformation provides a fermionic representation for SU(2)-invariant spin systems. The spin operators S^-_α and S^+_α transform as $S^-_\alpha \to T_{\alpha\beta}S^-_\beta$, $S^+_\alpha \to T^*{}_{\beta\alpha}S^+_\beta$, $\mathbf{T}\in\mathrm{SU}(2)$. The simplest SU(2)-invariant Hamiltonian corresponds to the XY model,

$$H = J\sum_{\mathbf{x},\hat{\boldsymbol{\mu}}}[S^+_\alpha(\mathbf{x})S^-_\alpha(\mathbf{x}+\hat{\boldsymbol{\mu}}) + S^-_\alpha(\mathbf{x})S^+_\alpha(\mathbf{x}+\hat{\boldsymbol{\mu}})] \quad (11)$$

($\hat{\boldsymbol{\mu}}$ runs over the unit cell vectors). Applying the mapping (4) and (5) one finds that the product $U(\mathbf{x})U^\dagger(\mathbf{x}+\hat{\boldsymbol{\mu}})$ becomes the link gauge field in the fermion hopping:

$$\begin{aligned}U(\mathbf{x})U^\dagger(\mathbf{x}+\hat{\boldsymbol{\mu}}) &= e^{i\sum_{\mathbf{z}\neq\mathbf{x}} j^{0a}(\mathbf{z})\omega^a(\mathbf{z}-\mathbf{x})} \\ &\quad\times e^{-i\sum_{\mathbf{z}\neq\mathbf{x}+\hat{\mu}} j^{0a}(\mathbf{z})\omega^a(\mathbf{z}-\mathbf{x}-\hat{\mu})} \\ &\equiv e^{i\sum_{\mathbf{z}} j^{0a}(\mathbf{z})W^a_\mu(\mathbf{z}-\mathbf{x})}\,. \end{aligned} \quad (12)$$

This defines the gauge potential $W^a_\mu(\mathbf{z})$, which can be computed in the continuum,

$$W^a_\mu(\mathbf{z}-\mathbf{x}) = -\epsilon^{\mu ab}\frac{(z-x)^b}{|\mathbf{z}-\mathbf{x}|^2} \qquad (\mathbf{z}\neq\mathbf{x})\,. \quad (13)$$

We identify W^a_μ as the potential of a monopole [14]. Thus, the XY Hamiltonian is mapped to a fermionic model, minimally coupled to an SU(2) non-Abelian gauge field:

$$H = J\sum_{\mathbf{x},\hat{\mu}}\psi^\dagger(\mathbf{x})\,e^{i\mathbf{A}_\mu(\mathbf{x})}\psi(\mathbf{x}+\hat{\boldsymbol{\mu}}) + H_G\,, \quad (14)$$

where $\psi(\mathbf{x}) = (\psi_1(\mathbf{x}),\psi_2(\mathbf{x}))$, and the SU(2) gauge field takes the form

$$\mathbf{A}_\mu(\mathbf{x}) = \sum_{\mathbf{z}\neq\mathbf{x}} j^{0a}(\mathbf{z})W^a_\mu(\mathbf{z}-\mathbf{x})\,. \quad (15)$$

In (14), H_G represents the Hamiltonian for the gauge field degrees of freedom. Its exact expression is not important to us here, so long as it contains SU(2)→U(1) symmetry–breaking interactions responsible for the presence of monopoles. The form of H_G depends on the model under consideration and on the physical significance one assigns to the gauge field [15].

One may view the non-Abelian gauge field $\mathbf{A}_\mu$ as a nondynamical artifact needed for the construction of the JW mapping. This point of view, however, would not lead to a local interaction between fermions ($\mathbf{A}_\mu$ would be just a new name for a nonlocal object). Alternatively, one may regard the $\mathbf{A}_\mu$ as a dynamical field whose classical equations possess a solution given by (15). This approach urges us to consider the SU(2) gauge symmetry as a true invariance of the physical system. In fact, the SU(2) gauge symmetry is not foreign to a spin system on the lattice. Any system of localized spins has a gauge symmetry which reflects the local freedom in the choice of the spin quantization axis. This phenomenon has been recently shown to give rise to a stability enhancement of the AF ordering in the Hubbard model. Also $\mathbf{A}_\mu$ is identified, in that model, as the field of magnonic excitations [16].

An additional term that could be included in the Hamiltonian is the analog of the Ising interaction, $S^z_{\alpha\beta}(\mathbf{x})S^z_{\beta\alpha}(\mathbf{x}+\hat{\boldsymbol{\mu}})$. This would generate a quartic nearest-neighbor interaction for the fermions,

$$\frac{1}{2}[1-\rho(\mathbf{x})][1-\rho(\mathbf{x}+\hat{\boldsymbol{\mu}})]+\frac{1}{2}j^{0a}(\mathbf{x})j^{0a}(\mathbf{x}+\hat{\boldsymbol{\mu}})\ . \qquad (16)$$

This includes, apart from the usual Ising form [the $\rho(\mathbf{x}) \times\rho(\mathbf{x}+\hat{\boldsymbol{\mu}})$ term], an additional (iso)spin current density interaction, $j^{0a}(\mathbf{x})j^{0a}(\mathbf{x}+\hat{\boldsymbol{\mu}})$. This issue will be discussed elsewhere.

Although our work deals strictly with spin systems in the lattice, it seems likely that the construction can be extended to the context of a 3+1 relativistic field theory. The operator U in that case may be related to the monopole creation operators studied by Marino and Stephany-Ruiz [17].

In the continuum, the statistical transmutation in the presence of monopoles is not new. Jackiw and Rebbi, and Hasenfratz and 't Hooft [18] have shown that in an SU(2) gauge theory isospin degrees of freedom can be converted into spin degrees of freedom in the field of a magnetic monopole. On the other hand, Goldhaber [19] has shown that if the system has odd-half integer isospin, a change in statistics is induced. This seems to be the reason behind the conspicuous presence of topological structures in the JW transformations.

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Note added.—It has been suggested to us that an analogous construction could be devised in which Skyrmions—rather than monopoles—induce the transmutation of statistics.

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