Effect of Betatron Motion on Particle Loss Due to Longitudinal Diffusion in High-Energy Colliders

Barry S. Newberger
Institute for Fusion Studies and Department of Physics, University of Texas-Austin, Austin, Texas 78712

James A. Ellison
Department of Mathematics, University of New Mexico, Albuquerque, New Mexico 87131

H.-Jeng Shih
Superconducting Super Collider Laboratory, Dallas, Texas 75237
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Strictly one-dimensional theories of particle loss due to longitudinal diffusion model the loss by an absorbing boundary condition at the separatrix of the underlying unperturbed motion. Particle loss always occurs at a physical aperture and the loss is always coupled to the betatron motion. A theory of particle loss which includes the effect of betatron motion is presented. Results are compared with Monte Carlo simulations.

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It has long been understood that noise in the rf systems of storage rings leads to a slow increase of the longitudinal emittance of the stored bunches [1-3]. Especially in the design of new hadron storage rings, intrinsic sources of noise must be identified and investigated to insure that they do not represent sources of unacceptable loss of beam lifetime [4,5]. Conversely, the issue of controlled injection of noise into the rf system of these machines for the purpose of extracting beam has recently become of considerable interest [6,7]. Here the idea is to "diffuse" particles in momentum until, due to dispersion, they reach a physical extraction septum.

Previous treatments of the loss process have been strictly one dimensional, taking no account of the betatron motion. The losses are described in terms of longitudinal action and are assumed to occur at the separatrix of the longitudinal motion. This does not correspond to a real aperture stop and the quantitative results from these theories exhibit significant discrepancies when compared, for example, to Monte Carlo tracking studies [7]. Knowledge of these loss rates is essential in the design of high-luminosity hadron colliders. In this paper, we obtain a statistical model of the transverse x coordinates of an ensemble of particles in a bunch, calculate the loss at an x septum, and find excellent agreement with the tracking results.

It is conventional [8] to describe the transverse motion in a synchrotron as a superposition of a closed orbit, x_{co}, and a (betatron) oscillation, x_β. The closed orbit, as the name implies, is periodic with the machine period; the betatron motion, in general, is not periodic. At a fixed lattice position, the closed orbit of a synchronous particle is the machine axis while an off-momentum particle executes an oscillation about the closed orbit position of the synchronous particle. This is the synchrotron oscillation, of frequency Ω for small amplitudes. The motion is on an energy oval in the longitudinal phase space of a simple pendulum. The position of the closed orbit is related to the relative momentum deviation, δp by $x_{co}=\eta\delta p$. Here, η is the dispersion which is a lattice function. Under certain assumptions, the effect of the noise can be described as a diffusion process in which there is a slow increase in amplitude of the synchrotron oscillation on a time scale long compared to a synchrotron period. Consequently, there is an increase in the closed orbit amplitude of these off-momentum particles. The position $x=x_{co}+x_\beta$ determines when a particle hits the stop. This could be an extraction septum, an array of collimators, or even a "dynamical" aperture which results from nonlinearities in the system.

The theory [9-11] of the longitudinal dynamics in a noisy rf system leads to a description of the evolution of an ensemble by a diffusion in the action, J, which is a constant of the unperturbed motion. The time scales of the diffusion in action, the synchrotron period, and the betatron oscillation period are disparate with $t_d \gg t_s \gg t_\beta$. Thus the collimation process sweeps an infinitesimal shell in the transverse phase space $(A, A+dA)$, where A characterizes a Courant-Snyder invariant and an infini-

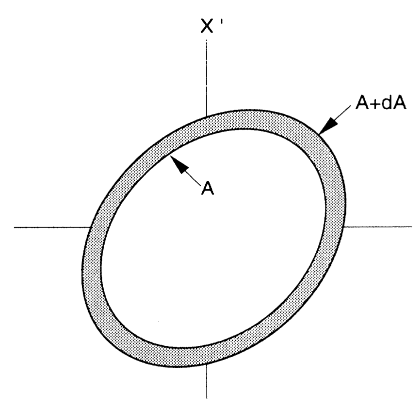

FIG. 1. Transverse phase space domain (schematic) swept by collimation process. The longitudinal phase space is similar.

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tesimal shell $(J, J+dJ)$ in the longitudinal phase space in a time t, $t_s \lesssim t$ (see Fig. 1). The maximum betatron displacement is related to A by $|x_\beta|_{\max} = \sqrt{\beta} A$, where β is the usual betatron function [8].

In the presence of an aperture stop, a particle slowly diffuses toward the periphery of the beam pipe until it strikes the stop. The horizontal position of the stop is x_c and we assume the "image" of the stop in momentum, x_c/η is inside the bucket. This is appropriate for either the proposed superslow extraction [7] or momentum scrapers [12]. The time it takes until the particle is lost, or conversely, the loss rate, depends on both the closed orbit position (equivalently, action) and the betatron displacement.

Consider particles which, at a given time, t, have not yet reached the stop. Referring to Fig. 2, we see that the protons which are inside the stop lie in the domain $\mathcal{S}$ in (J, x_β) space. The entire space is the quadrant $\mathcal{Q}$: $J, x_\beta \geq 0$. We have simply written x_β for $|x_\beta|_{\max}$ in order to keep the notation from becoming too cumbersome. The domain $\mathcal{S}$ is bounded by segments of the coordinate axes and the curve $\partial\mathcal{S}$: $\{x_\beta, J = J_b(x_\beta) | 0 \leq x_\beta \leq x_c\}$; $J_b(x_\beta)$ is the action for a particle on an energy surface, $k \equiv (\delta p)_{\max}/\delta p_s$, defined by $k = k_b(x_\beta) \equiv (x_c - x_\beta)/\eta \delta p_s$. Here δp_s is the relative bucket half-height. For the pendulum, $J_b(x_\beta)$ can be written in terms of elliptic integrals. It might be expected that, for these particles, the two random variables x_β and J might be statistically independent for all time because initially they are. That this is not so is the crucial observation in the construction of the theory. However, the calculation of the joint distribution is deceptively simple.

Consider the (time dependent) joint probability density $p(J, x_\beta, t)$ and the (time dependent) action conditional probability density conditioned on x_β, $p(J, t | x_\beta)$. Then, by definition,

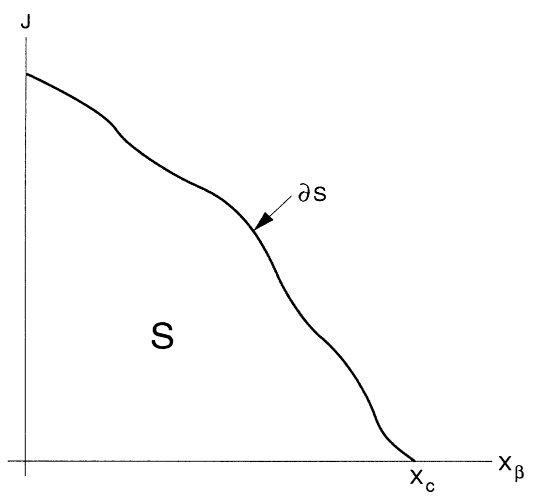

FIG. 2. Domain of random variables J and x_β for particles inside the aperture.

$$p(J, x_\beta, t) = p(J, t | x_\beta) p(x_\beta) , \tag{1}$$

where $p(x_\beta)$ is the marginal probability density of x_β. Notice that we have used the symbol p for three different functions. In what follows, we are careful to display the arguments so there should be no confusion over which function is used. These must be probability densities and must be defined throughout $\mathcal{Q}$. In the domain $\mathcal{S}$, the protons have a marginal x_β distribution which is just the initial distribution; they are Rayleigh distributed with parameter $\sqrt{\epsilon\beta}$, where ϵ is the emittance. (The transverse phase space variables are bi-Gaussian.) We must now find the conditional density, $p(J, t | x_\beta)$. This satisfies the same diffusion equation considered in earlier work [7,9-11]:

$$\frac{\partial}{\partial t} p(J, t | x_\beta) = \frac{\partial}{\partial J} \left[D(J) \frac{\partial}{\partial J} p(J, t | x_\beta) \right] , \tag{2a}$$

but with the boundary condition

$$p(J_b, t | x_\beta) = 0 . \tag{2b}$$

The action dependent diffusion coefficient is $D(J)$. The absorbing boundary now is dependent on x_β and is just the curve $\partial\mathcal{S}$ defined earlier. In $\mathcal{Q} - \mathcal{S}$, the joint density is a delta function. Furthermore, consistency requires that the identity

$$p(x_\beta) \equiv \int_0^\infty p(J, t | x_\beta) p(x_\beta) dJ \tag{3}$$

be satisfied. The joint density in $\mathcal{Q} - \mathcal{S}$ is constructed so this consistency condition is satisfied. It is never needed to evaluate any quantity of interest although it is implicit in the required conservation of probability. It should be noted that the same arguments apply if the marginal density $p(x_\beta)$ has time dependence provided the evolution is due to a process which is independent of the particle's longitudinal action, J. Finally, the fraction of protons which reach the stop is given, using conservation of probability, in terms of an integration of the joint density over the domain $\mathcal{S}$:

$$N(t) = 1 - \int \int_{\mathcal{S}} p(J, t | x_\beta) p(x_\beta) dJ \, dx_\beta . \tag{4}$$

It is also instructive to derive Eq. (4) in the context of a first passage problem. Suppose T is the (random) time a particle strikes the stop. This time depends on the particle's betatron motion and is a random variable. Conditioning on x_β, the law of total probability gives

$$P\{T > t\} = \int_0^{x_c} P\{T > t | x_\beta\} p(x_\beta) dx_\beta , \tag{5}$$

where the left hand side (lhs) is the probability of not reaching the stop in $[0, t]$. Now,

$$P\{T > t | x_\beta\} = P\{J(s) < J_b(x_\beta), 0 \leq s \leq t | x_\beta\} , \tag{6}$$

where $J(t)$ is the action stochastic process and the probability on the right hand side is $\int_0^{J_b} p(J, t | x_\beta) dJ$ where the

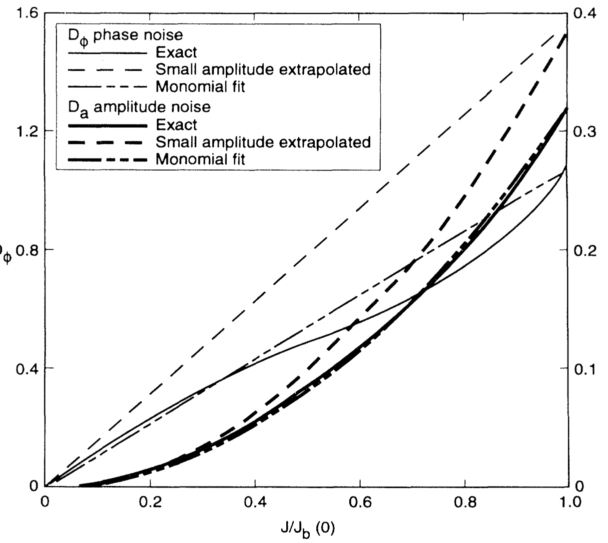

FIG. 3. Diffusion coefficients for amplitude and phase noise.

integrand is defined by Eq. (2). Therefore,

$$P\{T>t\}=\int_0^{x_c} p(x_\beta)\left(\int_0^{J_b(x_\beta)} p(J,t|x_\beta)dJ\right)dx_\beta\,, \quad (7)$$

and the lhs can be interpreted as the fraction of particles reaching the stop in the time interval [0,t]. This is the same as $1-N(t)$, Eq. (4).

In this paper, we consider two cases: white phase noise and white amplitude noise. The exact diffusion coefficients are shown in Fig. 3 and for small action are seen to be linear in the case of white phase noise and quadratic for white amplitude noise [11]. If one simply uses the small amplitude expressions for the diffusion coefficients all the way to $J_b(0)\equiv J_b(x_\beta)|_{x_\beta=0}$, these diffusion coefficients overestimate the actual diffusion. We see in Fig. 3 it is better to fit the approximations by forcing them to be equal to the actual diffusion coefficient at $J_b(0)$. The linear fit to the diffusion coefficient for white phase noise and the quadratic fit in the case of white amplitude noise are reasonable over most of the bucket; the approximation is very good in the latter case. This is important because the diffusion equation is easily solved by Fourier-Bessel series [9] for a linear diffusion coefficient and by Fourier transforms [13] for a quadratic diffusion coefficient.

It is straightforward to apply these solutions to our problem. We reiterate that the solution in terms of the absorbing boundary condition at $J_b(x_\beta)$ is the required conditional probability density. The solutions are functionals of the initial density. It is convenient to express the initial probability density in terms of k $(0\le k\le 1)$ rather than the action J. There is a one-one mapping between the two: $J(k)=(8\Omega/\pi)k^2B(k)$, where B is related to the complete elliptic integrals of the first and second kinds, K and E, respectively; $B(k)=1/k^2[E(k)-(1-k^2)K(k)]$ and $J_b(x_\beta)\equiv J(k=k_b(x_\beta))$. Then, the initial conditional probability density (in k) is

$$p(k,t=0|x_\beta)=(k/\sigma_k^2)\exp(-k^2/2\sigma_k^2)\,, \quad (8)$$

where $\sigma_k=\sigma_{\delta p}/\delta p_s$ and $\sigma_{\delta p}$ is the rms momentum spread. Note that because initially the random variables are statistically independent, the initial conditional density is the same as the initial marginal density.

In the case of a linear diffusion coefficient Eq. (4) gives

$$N(t)=1-2\int_0^{x_c} dx_\beta p_0(x_\beta)\sum_n \tilde{a}_n T_n(x_\beta,t)\,, \quad (9)$$

where $\tilde{a}_n=[J_1(j_n)X_b/j_n]a_n$, $X_b=J_b/J_s$, $J_s=8\Omega/\pi$, and $p_0(x_\beta)$ is the initial probability density of x_β. We have used the solution of Eq. (2):

$$p(k,t|x_\beta)=\sum_n a_n T_n(x_\beta,t)J_0(j_n k\sqrt{B(k)/X_b})\,, \quad (10a)$$

$$T_n(x_\beta,t)=\exp[-(\pi^3/16)(Q_s^2\sigma_\phi^2 j_n^2/X_b)t]\,, \quad (10b)$$

$$a_n=\frac{1}{X_bJ_1^2(j_n)\sigma_k^2}\int_0^{k_b} p(k,t=0|x_\beta)\times J_0(j_nk\sqrt{B(k)/X_b})dk\,, \quad (10c)$$

where j_n is the nth zero of the Bessel function J_0, t is the time measured in machine periods, T_0, $Q_s=(\Omega/2\pi)T_0$, and σ_ϕ (rad) is the phase noise standard deviation.

Likewise, in the case of a quadratic diffusion coefficient we use the Fourier integral solution [13] of Eq. (2) to find

$$N(t)=1-\frac{1}{2}\int_0^{x_c} dx_\beta p_0(x_\beta)\int_0^{k_b(x_\beta)} dk\, p(k,t=0|x_\beta)\left\{\operatorname{erfc}\left[\frac{\pi Q_s\sigma_a\sqrt{t}}{2}+\frac{\ln[k^2B(K)/X_b]}{2\pi Q_s\sigma_a\sqrt{t}}\right]-\frac{X_b}{k^2B(k)}\operatorname{erfc}\left[\frac{\pi Q_s\sigma_a\sqrt{t}}{2}-\frac{\ln[k^2B(K)/X_b]}{2\pi Q_s\sigma_a\sqrt{t}}\right]\right\}, \quad (11)$$

where σ_a is the standard deviation of the amplitude noise (relative to applied voltage) and erfc(z) is the complementary error function. Recall $p_0(x_\beta)$ is the Rayleigh distribution:

$$p_0(x_\beta)=(x_\beta/\epsilon\beta)\exp(-x_\beta^2/2\epsilon\beta)\,. \quad (12)$$

We will use these results to make a comparison with Monte Carlo tracking studies [7,11]. However, one point needs to be discussed. This is to account for what we call transient particles. These are particles whose initial phase coordinates are such that they will strike the aperture stop in a synchrotron period independent of the diffusion. They contribute to the count in Eqs. (9) and

TABLE I. Values of N_h. The two theoretical values are for the case of small amplitude extrapolation and monomial fit.

	Phase noise			Amplitude noise	
σ_ϕ	N(sim)	N(theory)	σ_a	N(sim)	N(theory)
0.2	584	732/595	0.5	745	772/718
0.2[a]	343	542/363	0.2	182	211/178
0.1	187	261/179	0.1	67	56/48
0.05	51	69/49	0.05	22	19/17
0.02	13	14/14	0.02	7	7/6

[a] $\beta_x = 346$ m.

(11) as well as in the simulations. The number of transient hits are simply found by integrating the initial joint distribution in x_{co}, x_β over the appropriate domain in the space of the random variables. Initially, the random variables x_{co} and x_β are independent and the initial joint distribution is just the product of the initial distributions of the individual random variables. We do not present the details; the calculation is completely straightforward.

We have been using Monte Carlo tracking studies in the linear Superconducting Super Collider (SSC) lattice to simulate the effect of noise in the rf system for the purpose of extracting particles [7,11] as well as understanding its impact on collider performance [5,11]. Typically, the simulations follow 1000 particles for about 10^6 turns around the ring. In both the phase noise and amplitude noise cases, independent random perturbations, Gaussian with variance σ^2, are applied at each turn. Comparisons of the numbers of tracks reaching the aperture at the end of the run with the value of N from the appropriate expression above are summarized in Table I.

The simulation results are from a single realization of the random process. The machine parameters are the nominal values for the SSC collider rings, $\sigma_{\delta p} = 5 \times 10^{-5}$, $\epsilon_x = 1\pi$ mm mrad, $\Omega = 26.6$ rad/sec, and $\delta p_s = 2.6 \times 10^{-4}$. The value of β_x at the aperture stop was 1385 m in all cases but one. The exceptional case is indicated. The dispersion $\eta = 4m$. For both simulation and theory, 31 transient particles have been taken into account. The noise variances are larger than would be encountered in practice. This is necessary to get a measurable loss rate in a reasonable computing time. The first figure in the column of theoretical values is the result using the extrapolated small amplitude diffusion coefficient and the second value was obtained by using the diffusion coefficient fit as described earlier. The agreement between the simulation and theory is generally very good for the latter. The largest discrepancy, about 30%, occurs for the amplitude noise case with $\sigma_a = 0.1$. There is reason to believe it might be due to statistical fluctuations in the single realization of initial conditions.

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