Bose Condensation and Small-Scale Structure Generation in a Random Force Driven 2D Turbulence

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We consider two-dimensional flow stirred by a small-scale, white-in-time random noise in the zero viscosity limit. Numerical simulations show that, after a transient state, an inertial-range energy spectrum $E(k) \propto k^{-x}$ with $x=\frac{5}{3}\pm 0.05$ is established by the inverse cascade process. This range grows in time until a Bose condensate is formed at the largest scales in the system ($k \approx 1$). Prior to condensate formation the statistics of velocity differences are extremely close to Gaussian, and only after Bose condensation strong deviations from Gaussian statistics are detected at small scales. The structures responsible for this effect are identified.

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High Reynolds number turbulent flows are characterized by complexity, loosely defined as an intricate interplay of coherent and random dynamics. This makes the theoretical description of strong turbulence a major scientific challenge. The existence of coherent structures living in a random, structureless background has been demonstrated in numerous numerical and physical experiments. Their role is manifested in strong deviations from Gaussian values of the probability density functions (PDFs) of various turbulence quantities such as $P(\Delta u(r))$, where $\Delta u(r)=u_i(\mathbf{x}+r\hat{\mathbf{i}})-u_i(\mathbf{x})$ and u_i is the ith component of velocity $\mathbf{u}(\mathbf{x})$. Although the dynamics of the structures in turbulence have been a subject of intense interest during the last decade, the mechanisms governing their formation and behavior are not known. The problem is that a typical experiment, numerical or physical, deals with statistically steady flows, characterized by several external parameters: The power of the energy source ϵ and the wavelengths k_s, k_i, k_0, and k_d corresponding to the system size, integral scale, energy-input scale, and dissipation scale, respectively. To stabilize the flow, a viscous or other dissipation mechanism is always assumed for compensation of the energy input. It is clear that strong nonlinearity is necessary for structure generation. However, the following question must be asked: Is the nonlinear term solely responsible for strong deviations from Gaussian statistics, or do these deviations result from the combined action of nonlinearity and external parameters? Another question to be addressed is: What is the role of the largest, box-size eddies in coherent structure formation? This question is not simple since these eddies can, in principle, be responsible for symmetry breaking, the dynamical consequences of which are not addressed by a typical analytical theory of turbulence. In this Letter we attempt to answer some of these questions for the example of two-dimensional (2D) flow driven by a white-in-time random force.

We consider the Gaussian noise defined by the correlation function

$$\langle f_i(\mathbf{k},t)f_j(\mathbf{k}',t')\rangle \propto 2\epsilon\frac{\delta(k^2-k_0^2)}{k^{d-2}}\delta(\mathbf{k}+\mathbf{k}')\delta(t-t') = D(k)\delta(t-t')\delta(k^2-k_0^2)\,. \qquad (1)$$

Energy conservation following the Navier-Stokes equations leads to

$$-J(k) = -2\nu\int_k^\infty z^2E(z)dz + \epsilon\int_k^\infty \delta(z^2-k_0^2)dz^2 - \frac{\partial K}{\partial t}\,, \qquad (2)$$

where $J(k)$ is the energy flux and $K(k)=\int_k^\infty E(z)dz$. Equation (2) satisfies the natural boundary condition $J(k)\to 0$ when $E(k)\to 0$ fast enough as $k\to\infty$. In the 2D case there exists the additional law of enstrophy conservation. It is usually assumed that enstrophy conservation leads to a state of constant enstrophy flux J_Ω for $k>k_0$. This gives an energy spectrum in this region $E(k)\approx k^{-x}$ with $x>3$. Kraichnan's theory [1] yields $x=3$ with logarithmic corrections, Saffman [2] proposed $x=4$, and the recent conformal theory by Polyakov [3] gives $x=\frac{25}{7}$, all satisfying the criterion governing the direction of the enstrophy flux toward small scales. It can also be shown that the energy flux in this range is equal to zero. On the other hand, in the interval $k<k_0$, the assumption of constant energy flux leads to the Kolmogorov spectrum. Relation (2) gives that $J(k)$ is a *negative* constant, and thus that energy flows from small to large scales (inverse cascade).

In what follows, we wish to reduce the number of external parameters that may be responsible for breaking the symmetries of the Euler equations. Therefore we consider scaling solutions in the limit of zero viscosity so that k_d is eliminated from the problem. Since $\int_0^\infty J(k) =0$, one sees from (2) that a steady state is impossible when $\nu=0$ because the total energy $E(k,t)=\epsilon t\to\infty$ for $t\to\infty$. In this case, the most general self-similar solution is

$$E(k,t) = C_K\epsilon^{2/3}k^{-5/3} \times \left[\frac{k}{k_i(t)}\right]^{-x+5/3}\left[\frac{k}{k_0}\right]^{-\gamma}\phi\left[\frac{k}{k_i(t)}\right], \qquad (3)$$

where the integral scale $L(t)\propto k_i^{-1}(t)$ and the dimensionless scaling function $\phi(z)\to 1$ for $z\gg 1$ and $\phi(z)\to 0$ for $z\ll 1$. We will assume that the variation of $\phi(z)$ from

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0 to 1 takes place in a narrow range $z \approx 1$, so that $\phi(z)$ is close to a step function. It also follows from the continuity equation (2) that $J(k) = -\epsilon\psi(k/k_i)\theta(-k+k_0)$, with $\theta(z)=0$ for $z<0$ and $\theta(z)=1$ for $z>0$. In the interval $k_i(t)<k<k_0$, $\phi(z)=\psi(z)=1$. Integrating (3) leads to $E(t)=\epsilon t = C_1\epsilon^{2/3}k_0^{\gamma}k_i^{-2/3-\gamma}(t)$, where $C_1 = C_K \times\int_1^\infty z^{-x-\gamma}\phi(z)dz$. Solving for $k_i(t)$ gives

$$k_i(t) = \left(\frac{C_1 k_0^{\gamma}}{\epsilon^{1/3}t}\right)^{1/(2/3+\gamma)}, \tag{4}$$

which is similar to a relation derived by Kraichnan [1]. Assuming the existence of a stationary solution at $k \gg k_i(t)$, relation (3) gives $x=\frac{5}{3}$. The value of the exponent γ must be determined from dynamical theory. When $k_i(t)\rightarrow k_s$, energy accumulation leading to a steepening of the energy spectrum is expected at the largest scales which can, in principle, lead to Bose condensation, characterized by the energy spectrum $E(k,t) = b(t)\delta(k-1)+C_K\epsilon^{2/3}k^{-x}$, where C_K is the "Kolmogorov" constant. In this case, a quasisteady state with constant energy flux at the scales $k_0 \gg k \gg 1$ can be maintained and the energy spectrum $E(k)$ at these scales can be close to the Kolmogorov spectrum. The possibility of Bose condensation in 2D turbulence was discussed in an extraordinary work by Kraichnan [1], and confirmed numerically [4]. In 128^2 simulations, Frisch and Sulem [5] observed the inverse energy cascade and concluded that the exponent of the energy spectrum was close to the Kolmogorov value $x=\frac{5}{3}$. However, the statistical properties and the dynamics of Bose condensation were not addressed.

To investigate the details of the dynamics in the inverse cascade regime, we conducted direct numerical simulations of 2D flow in the limit of zero viscosity, driven by a white-in-time Gaussian force. The simulations were performed on the Intel Hypercube and Delta using a pseudospectral code with 2048^2 Fourier modes to calculate the stream function $\Psi(\mathbf{k},t)$ in a periodic square, where the velocity $\mathbf{u}(\mathbf{k},t)$ is given by $u_1 = -ik_2\Psi$ and $u_2 = ik_1\Psi$. The force was localized in the interval $500 \le k_0 \le 525$, and the superviscous dissipation term $D = 6.1\times10^{-47}k^{16}\mathbf{u}$ was used (in place of $\nu k^2\mathbf{u}$) to stabilize vorticity production in the region $k>600$. It has been thoroughly checked that the dissipation term was negligibly small everywhere in the inverse cascade interval $500<k$. The initial condition $\mathbf{u}(\mathbf{k})=0$ was used in all our simulations. The results of the 2048^2 run are presented in Figs. 1 and 2.

Figure 1 shows the time evolution of the energy spectrum. The early-time spectral distribution is $E(k,t) \approx k^0$, corresponding to weak nonlinearity. At intermediate times, the Kolmogorov spectrum $E(k)\approx k^{-x}$ with $x\approx\frac{5}{3}$ is established in the interval $k_i(t)<k<k_0$. As predicted, the energy flux $J(k)$ is a negative constant for $k_i(t)<k<k_0$ and the enstrophy flux $J_\Omega(k)=0$ for $k<k_0$. At all times the energy flux $J(k)$ satisfied (2) and the energy spectrum was close to the prediction (3) with $x=\frac{5}{3}$. Figure 2 exhibits the compensated energy spectrum $k^{5/3}E(k)$ at the last time in Fig. 1, representative of the intermediate times with spectrum $E(k)\propto k^{-x}$, $k_i(t)<k<k_0$. Using different exponents x we were able to establish $x=\frac{5}{3}\pm0.05$ with Kolmogorov constant $C_K\approx7.0$. The most striking feature of the flow is demonstrated in the inset of Fig. 2, showing $S_2\equiv\langle(\Delta u)^2\rangle$ and the normalized even moments $F_{2n}\equiv\langle(\Delta u)^{2n}\rangle/\langle(\Delta u)^2\rangle^n$, $n=2$–4. The odd moments are small and will be discussed in a future publication. The value of dis-

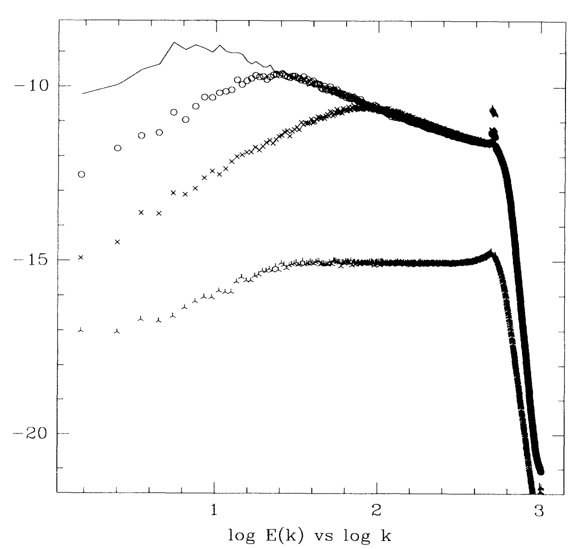

FIG. 1. Time evolution (increasing upward) of $E(k)$ for the 2048^2 run.

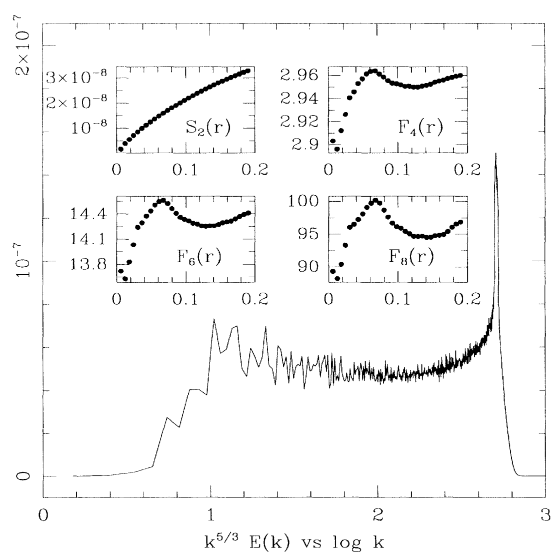

FIG. 2. Compensated spectrum $k^{5/3}E(k)$ for the last time (solid curve) in Fig. 1. Inset: The second moment of velocity S_2 and the normalized moments F_{2n}, $n=2$–4.

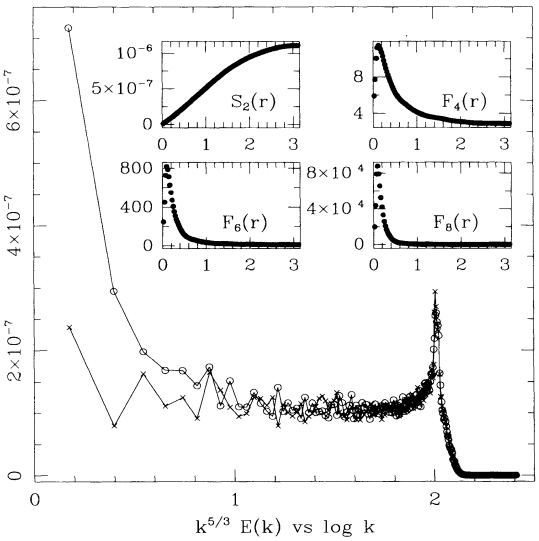

FIG. 3. Compensated spectra at an early time t_1 (crosses) and a later time t_2 (circles) after condensate formation. Inset: S_2 and F_{2n}, $n=2$–4 at t_2.

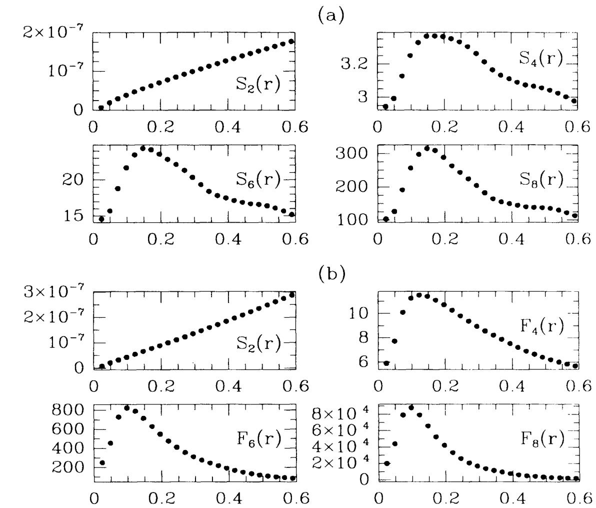

FIG. 4. S_2 and F_{2n}, $n=2$–4 at (a) t_1 and (b) t_2. The range of r is reduced to emphasize the region of intermittency.

placement r corresponding to the input of energy is $r=0.01$. One sees that the normalized even moments have a near Gaussian distribution $P(\Delta u)$. Thus there is no intermittency of Δu in this high Reynolds number flow. The same result has been obtained for two additional simulations differing by the values of ϵ and k_0.

For later times when $k_i(t) \to k_s = 1$, the time development of the large scales became prohibitively slow in the 2048^2 simulation, and therefore it was necessary to repeat the experiment with the lower resolution of 512^2 Fourier modes. The 512^2 simulation was forced in the interval $100 \leq k_0 \leq 105$ and the superviscous dissipation was $D=4.0\times10^{-36}k^{16}\mathbf{u}$. Figure 3 displays compensated spectra for two times t_1 (crosses) and t_2 (circles) after $k_i(t)=1$, and clearly indicates the formation of the Bose condensate. The inset of Fig. 3 shows S_2 and F_{2n}, $n=2$–4 for t_2, where the energy-input scale is $r=0.06$. One sees that the values of the normalized even moments remain the Gaussian values for $r>1$. However, for small values of the displacement $r<0.6$, the moments deviate strongly from the Gaussian values, indicating intermittency at small scales. In Fig. 4, the range of r has been reduced to emphasize the region of intermittency. Figures 4(a) and 4(b) correspond to the times t_1 and t_2, respectively, and show that small-scale intermittency increases with time as the energy in wave number $k=1$ increases.

In physical space, the observed distribution of vorticity is structureless before the formation of the condensate. In the condensate state, as energy piles up in wave number $k=1$, the vorticity localizes in space until it is eventually concentrated in two vortices of opposite sign. The vorticity distribution inside of each vortex is given by $\omega \approx \omega_0(t)e^{-ar}$ with $a \approx 2\pi/50$, which is approximately twice the characteristic scale $2\pi/k_0$ of the force. The amplitude of $\omega_0(t)$ grows in time, while $\omega_{\rm rms}$ stays approximately constant. Although these vortices move randomly in space, the distance L_d between their centers stays approximately constant and is equal to $L_d \approx \pi$ (half of the box size). Thus Bose condensation leads to the generation of a structure (a vortical "dipole") characterized by the two length scales L_d and a^{-1}. Figure 5(a) shows the

(a)

(b)

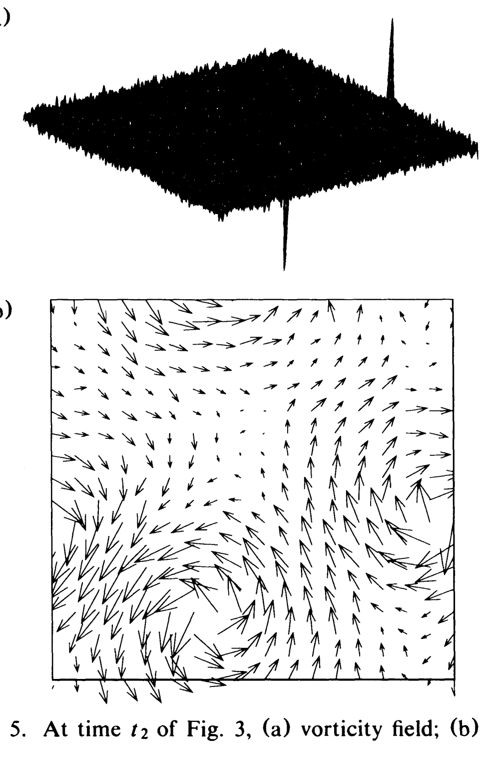

FIG. 5. At time t_2 of Fig. 3, (a) vorticity field; (b) velocity vectors.

three-dimensional vorticity field for the latest time reached in the 512^2 simulation. The maximum value of vorticity increased by a factor of approximately 5 from the time of condensate formation to the final time of our simulation and reached the value $\omega_{\rm max} \approx 25\omega_{\rm rms}$. Figure 5(b) shows the velocity field corresponding to these vortices.

To conclude, two unexpected phenomena were observed in our numerical experiments. First, although the energy spectrum prior to the Bose condensate formation is described by the Kolmogorov relation corresponding to strong nonlinearity, the statistics of velocity differences do not deviate from Gaussian values (see Figs. 1 and 2). We do not have a theoretical explanation of this fact, though some dynamical considerations are as follows. Let us assume that coherence is established as a result of interactions between the small-scale eddies of size r and velocity fluctuations of the integral scale $L(t) \propto k_i^{-1}(t)$. The characteristic time of the process is of the order of the eddy-turnover time of the largest scales $\tau_i \propto k_i^{-2/3-\gamma}$. However, according to (4), this is exactly the time scale of significant variation of k_i. The even larger eddies, generated as a result of the inverse cascade, tend to reduce the efficiency of coherence generation. Only after long-living large-scale flow features are created as a result of condensation can small-scale structures be formed. The second unexpected result is that these small-scale structures are generated at scales of the order of the input scale k_0. We can see from Fig. 3 that at all scales $l \gg k_0^{-1}$, the deviations from Gaussian statistics are negligibly small even after formation of the condensate. Moreover, as can be judged from Fig. 4, the range of scales where the deviations are largest becomes narrower with time, concentrating in the vicinity of $k \approx k_0$. At present we can only speculate about the dynamics of the structure generation and the significance of the input scale. The supercondensation processes considered in [6], dealing with the dynamics of an equilibrium ensemble of point vortices, might give some clues to understanding the results found in this paper.

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