Ohm's Law for a Relativistic Pair Plasma

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We derive the fully relativistic Ohm's law for an electron-positron plasma. The absence of nonresistive terms in Ohm's law and the natural substitution of the four-velocity for the velocity flux in the relativistic bulk plasma equations do not require the field gradient length scale to be much larger than the lepton inertial lengths, or the existence of a frame in which the distribution functions are isotropic.

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For a plasma, Ohm's law describes the relation between the induced current and the plasma electric field. For an ion-electron plasma, the field depends on resistive, inertial, and Hall effect contributions. The result is usually derived for the nonrelativistic limit in the Boltzmann picture [1]. Ohm's law plays a direct role in the magnetic induction equation used in the description of bulk plasma dynamics.

Relativistic plasma models have been effective in explaining the observations of relativistic astrophysical jets and winds [2]. Such models have generally employed the relativistic continuity equation as a vanishing four-divergence of the bulk four-velocity, and Ohm's law as a simple bulk four-vector generalization of the nonrelativistic equation [2,3].

We shall see that for a two-component relativistic plasma composed of different mass particles, the natural use of these magnetohydrodynamic (MHD) forms for the continuity equation and Ohm's law requires the existence of a reference frame in which both distribution functions are isotropic in momentum. The constraint results from the nonlinear relation between momenta and velocities in the relativistic regime.

This requirement is nontrivial because distribution function isotropy also requires the plasma under study to be microinstability saturated; otherwise microinstabilities could grow because of distribution function anisotropy. Yet, evidence for the presence of anisotropies and microinstabilities in relativistic winds has come from observations of the Crab Nebula [4]. In particular, the Weibel instability has been suggested to explain the presence of wisps downstream from relativistic shocks [5]. Anisotropies downstream from relativistic shocks are likely present in jets as well [6].

Depending on density and temperature conditions, anisotropies in relativistic plasmas may also arise from anisotropic radiation onto a plasma, such as in coronae of stars or in models active galactic nuclei (AGN) for which a pair atmosphere forms [7]. The impinging of winds and jets onto ambient media also produces anisotropies [8].

In this paper we show that in contrast to an ion-electron plasma (1) only a resistive contribution to Ohm's law for a relativistic e^+-e^- plasma is relevant under quite general conditions and (2) anisotropy in the distribution functions need not affect the form of the unperturbed relativistic bulk equations for the pair plasma. Thus, relativistic bulk dynamics for pair plasmas which exhibit evidence for microinstabilities are appropriately described by the relativistic MHD formalism, whereas ion-electron plasmas which exhibit such instabilities are not.

The Boltzmann equation is given by

$$\partial f/\partial t + v^i\partial_i f + F^i\partial_{p_i} f = [\Delta_t f]_{\rm coll} + [\Delta_t f]_{\rm radiat} + [\Delta_t f]_{\rm creation} + [\Delta_t f]_{\rm annihilation}\,, \tag{1}$$

where v^i is the particle velocity, p^i is the particle momentum, t is the time, F^i is the electromagnetic force (since we ignore gravity), $f=f(\mathbf{x},\mathbf{p},t)$ is the scalar distribution function, and the terms on the right are schematic. The cross sections present in the last two terms are approximately equal for relative lepton velocities $\sim c$ [9], and it is reasonable to assume these contributions are in equilibrium. We shall also assume that collisional losses dominate synchrotron radiation losses, which is acceptable for [10]

$$d|\mathbf{p}|/dt|_{\rm synch}/|e\mathbf{v}\times\mathbf{B}| \ll 10^{-16}\gamma^2 B\sin\phi\,, \tag{2}$$

where γ is the particle Lorentz factor, ϕ is the pitch angle, e is the positron charge, and B is the magnetic field measured in gauss.

Define the plasma quantities, number density,

$$n_\pm = \int f_\pm(\mathbf{x},\mathbf{p},t)\,d^3p\,, \tag{3}$$

and velocity flux density,

$$\phi^i_\pm = \int v^i_\pm f_\pm(\mathbf{x},\mathbf{p},t)\,d^3p = n_\pm\langle v^i_\pm\rangle\,. \tag{4}$$

Adopting the signature $(+,-,-,-)$, we then have the flux density four-vector $\phi^\mu_\pm=(cn_\pm,\phi^i_\pm)$ and current density four-vector $j^\mu=e(\phi^\mu_+-\phi^\mu_-)$. The energy and momentum densities are the (00) and (i0) components of the symmetric kinetic tensor given by

$$K^{ik}_\pm = \int v^i_\pm p^k_\pm f_\pm\,d^3p = n_\pm\langle v^i_\pm p^k_\pm\rangle \tag{5}$$

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and

$$K^{0\mu}_{\pm}=(\epsilon_{\pm},c\Pi^{i}_{\pm}), \tag{6}$$

where $\Pi^i_{\pm}$ is the momentum density of either positrons or electrons.

Quantities without the $\pm$ shall refer to the plasma as a whole—the sum of the component contributions. We require the existence of a proper frame moving with four-velocity U_μ in which the charge density, the velocity flux density, and the momentum density vanish. We denote quantities in this frame with a superscript asterisk. For an e^+-e^- plasma this means

$$j_0^*=e(n_+^*-n_-^*)=0, \tag{7a}$$

$$\phi^{i*}=0, \tag{7b}$$

and

$$\Pi^{i*}=m_e(n^*/2)(\langle\gamma_+v_+^i\rangle^*+\langle\gamma_-v_-^i\rangle^*)=0. \tag{7c}$$

Since $K_{\mu\nu}$ has ten independent components, we can write

$$K^*_{\mu\nu}=P^*_{\mu\nu}+A^*_{\mu\nu}, \tag{8}$$

where the symmetric tensors $P_{\mu\nu}$ and $A_{\mu\nu}$ satisfy

$$P^*_{ij}=P\delta_{ij},\quad P_{0i}=0,\quad P_{00}=\epsilon^*, \tag{9}$$

and

$$A^*_{0\mu}=0. \tag{10}$$

In (8)–(10), P is the scalar pressure and A^*_{ij} has five independent components that measure anisotropy.

Finally, define the four-vector H^μ by

$$H^\mu=[2/(m_++m_-)][\rho^*U^\mu-(m_+\phi^\mu_++m_-\phi^\mu_-)], \tag{11}$$

where ρ^* is the proper frame rest mass density. For a pair plasma this becomes

$$H^\mu_{\rm pair}=(1/m_e)[\rho^*U^\mu-m_e\langle\phi^\mu\rangle]. \tag{12}$$

Thus from (7b), $H^{i*}_{\rm pair}$ vanishes. But $H^{0*}_{\rm pair}=0$ by definition, so $H^{\mu*}_{\rm pair}=0$. Since $H^\mu_{\rm pair}$ is a four-vector, the vanishing of $H^{\mu*}_{\rm pair}$ implies that $H^\mu_{\rm pair}=0$ in all frames. Note that $H^{\mu*}$ measures the heat flux density per unit mass in the proper frame, so we shall call H^μ the heat flux density four-vector.

The procedure used to derive Ohm's law for a relativistic pair plasma is as follows: (i) First we obtain a "resistive" type collision term appropriate for a nearly collisionless relativistic pair plasma. (ii) Second, we relate this to the current density. (iii) Finally, a subtraction of the momentum density equations for the positrons and electrons yields the desired result.

We can find the value of the momentum density in any frame by Lorentz transforming the stress-energy tensor. The result is

$$\begin{aligned}c\Pi^i&=c^2m_e(n_+\langle u^i_+\rangle+n_-\langle u^i_-\rangle)\\&=\gamma_V^2(P+\epsilon^*)V^ic^{-1}+A^{0i},\end{aligned} \tag{13}$$

where u^i is a spatial component of the particle four-velocity, V^i is a component of the bulk three-velocity, γ_V is the bulk Lorentz factor, and the anisotropy term is given by

$$A_{0i}=A^*_{ij}U^j+U^kU^lA^*_{kl}U^i/(\gamma_V+1). \tag{14}$$

Note that since the proper frame four-vector $A^{\mu\nu*}U^*_\nu=0$, we know that it is zero in all frames. Thus $A_{i0}=A_{ik}U^k$ and (13) can be written

$$\begin{aligned}c\Pi^i&=c^2m_e(n_+\langle u^i_+\rangle+n_-\langle u^i_-\rangle)\\&=\gamma_V(P+\epsilon^*)U^i+A^{ij}U_j.\end{aligned} \tag{15}$$

Inverting (15) we obtain

$$\begin{aligned}\gamma_VV^ic^{-1}=U^i=[&c^2m_e(n_+\langle u^i_+\rangle+n_-\langle u^i_-\rangle)\\&-A^{ij}U_j]/\gamma_V(P+\epsilon^*).\end{aligned} \tag{16}$$

The average four-momentum gains from collisions in the proper frame are given by

$$\Delta p^{\mu*}_+=-\Delta p^{\mu*}_-=(m_e/2)(\langle u^\mu_+\rangle^*-\langle u^\mu_-\rangle^*). \tag{17}$$

The approximate proper frame pair plasma collision term is then

$$\begin{aligned}(n^*v_c^*/4)\Delta p^{\mu*}_+&=-(n^*v_c^*/4)\Delta p^{\mu*}_-\\&\equiv P^{\mu*}_{+-}=-P^{\mu*}_{-+},\end{aligned} \tag{18}$$

where v_c^* is the proper frame collision frequency.

Equations (12) and (16) give

$$\begin{aligned}m_ec^3(n_+\langle u^i_+\rangle+n_-\langle u^i_-\rangle)-cA^{ij}U_j=\gamma_V(P+\epsilon^*)n^{*-1}&\\\times(\phi^i_-+\phi^i_+).&\end{aligned} \tag{19}$$

Now in the electron frame, we have

$$c^2\Pi^{i(-)}_+-[cA^{ij}U_j]^{(-)}=\gamma_V^{(-)}(P+\epsilon^*)n^{*-1}\phi^{i(-)}_+, \tag{20}$$

and in the positron frame

$$c^2\Pi^{i(+)}_--[cA^{ij}U_j]^{(+)}=\gamma_V^{(+)}(P+\epsilon^*)n^{*-1}\phi^{i(+)}_-. \tag{21}$$

Transforming (20) and (21) to the proper frame, subtracting, and using (7) and (18) gives

$$v_c^*(\Pi^{i*}_+-\Pi^{i*}_-)=2P^{i*}_\pm=n^*e\eta_rj^{i*}, \tag{22}$$

where the effective resistivity η_r is given by

$$\begin{aligned}\eta_r=v_c^*(n^*e)^{-1}c^{-2}\{&(P+\epsilon^*)(n^*e)^{-1}[\gamma_c^{*2}(\gamma_c^*+1)^{-1}j^{\mu*}j_{\mu*}(en^*c)^{-2}+\gamma_c^*+1]\\&-\gamma_c^*\epsilon^*(en^*)^{-1}-2v_c^{*-1}\gamma_c^{*2}(\gamma_c^*+1)^{-1}(n^*e)^{-2}j^*_\mu P^{\mu*}_\pm\},\end{aligned} \tag{23}$$

and γ_c^* is the Lorentz factor corresponding to the velocity

$$(n^*/2)^{-1}\phi_+^{i*} = -(n^*/2)^{-1}\phi_-^{i*} = j^{i*}/en^* . \tag{24}$$

We have used the four-vector indices in η_r since $j_0^* = 0$.

Equation (22) suggests that we subtract the momentum equations for the electrons and the positrons to obtain Ohm's law. This is standard in the nonrelativistic case, but is only fruitful in the relativistic case because $H^\mu_{\mathrm{pair}} = 0$.

The ith component of the proper frame relativistic energy momentum tensor for positrons, as obtained from the first moment of the Boltzmann equation, is given by

$$\partial_0 \Pi_+^{i*} = -\partial_k K_+^{ki*} - e(n^*/2)E^{i*} - [e\langle \mathbf{v}_+/c\rangle \times \mathbf{B}]^{i*} + P_{+-}^{i*} . \tag{25}$$

Subtracting the analogous equation for electrons, and using (8), (18), and (22) we get

$$E^{*i} = \eta_r j^{i*} - m_e c(2e\rho^*)^{-1}\partial_k (A_+^{ki*} - A_-^{ki*}) - \partial_0[(\eta_r/v_c^*)j^{i*}] . \tag{26}$$

The requirements (7a) and (7b) which led to (26) are assumptions about the zeroth and first moments of the Boltzmann equation. If we further assume that

$$A_+^{ij*} = A_-^{ij*} , \tag{27}$$

the second term on the right-hand side of (26) would vanish. Then, in the steady state, we would be left with

$$E^{*i} = \eta_r j^{i*} . \tag{28}$$

Note that the dependence on the current in (28) results from (22). The latter follows, for example, even in the Fokker-Planck approximation if, as in our case, the charge density vanishes in the proper frame. In addition, note that none of the assumptions that led to (28) require the existence of a reference frame in which the distribution functions of positrons and electrons are isotropic.

In a general frame, (28) becomes

$$F^{\mu\nu}U_\nu = \eta_r j^\mu + j^\tau U_\tau U^\mu , \tag{29}$$

where we have added a convection term. Equation (29) resembles the magnetofluid relation [11]. Here, however, the effective resistivity η_r depends on a scalar function of the current and momentum density of the plasma components. This dependence is eliminated when we choose the coordinate axes such that the proper frame velocity lies on a principal axis. In addition, γ_c^* factors out of (23) and we have

$$\eta_r \to \eta_r^{(\mathrm{pr})} = v_c^* c^{-2}(n^*e)^{-2}\{2(P+\epsilon^*) - \epsilon^*\} = v_c^* c^{-2}(n^*e)^{-2}(2P+\epsilon^*) , \tag{30}$$

where the superscript (pr) indicates that the current flows along a principal axis.

Note that unlike the Ohm's law for an ion-electron plasma, there are no Hall effect or pressure contributions to (29). The vanishing of the bulk Hall effect term is the result of an effectively zero net particle gyration frequency; the sum of the electron and positron contributions vanishes due to opposite streaming of positrons and electrons around the field lines. The vanishing of the pressure term results because our assumptions have appealed to the mass symmetry of the problem, eliminating diffusion. The Hall effect and pressure terms are negligible for an ion-electron plasma only when $\lambda_{\mathrm{fg}} \gg \lambda_i$ and $\lambda_{\mathrm{fg}} \gg \lambda_i \beta$, respectively, where λ_{fg} is the length scale of the field gradients, λ_i is the ion-inertial length, and β is the ratio of thermal to magnetic pressure.

The vanishing of the heat flux, which led to (22), is also the condition which allows substitution of the bulk four-velocity components for the bulk four-flux components in the plasma continuity equation. The point is that the usual integration of (1) over momentum gives $\partial_\mu \phi^\mu = 0$, so substitution of the four-velocity for the four-flux ϕ requires $H^\mu = 0$. If $H^\mu \neq 0$, then this substitution would produce an inhomogeneous equation. Because $m_+ \gg m_-$, for a relativistic ion-electron plasma, $H^{\mu *}$ only naturally vanishes if both distribution functions are isotropic in the proper frame. This is true whether or not the proton component has a relativistic temperature.

Note that H^μ vanishes for *any* two-component plasma in the nonrelativistic limit. To see this note that

$$H^{i*} \propto -m_+ n_+^* \langle v_+^i \rangle^* + -m_- n_-^* \langle v_-^i \rangle^* = m_+ n_+^* \langle \gamma_+ v_+^i - v_+^i \rangle^* + m_- n_-^* \langle \gamma_- v_-^i - v_-^i \rangle^* , \tag{31}$$

where the second equality follows since the proper frame momentum density vanishes by definition. Thus when $n_+^* = n_-^*$, (31) vanishes when $\gamma_+ = \gamma_- = 1$. Therefore H^μ vanishes by the argument that follows Eq. (12).

Finally, note that our proper frame result is not the full nonrelativistic limit, because of the relativistic temperature. In the nonrelativistic limit, we have the additional result that the pressure drops out of (23) and that $\epsilon^* = n^* m_e c^2$ so

$$\eta_r^{(\mathrm{pr})} \to \eta = v_c n^* m_e c^2/(n^* ec)^2 \tag{32}$$

$$= m_e v_c/n^* e^2 = 2\mu_{\mathrm{pair}} v_c/n^* e^2 , \tag{33}$$

where η is the nonrelativistic resistivity and $\mu_{\mathrm{pair}} \equiv m_e/2$ is the reduced mass for the pair plasma. Equation (33) is the usual nonrelativistic result. In the case of a nonrelativistic ion-electron plasma, μ_{pair} is replaced by $\mu_{ie} \equiv m_i m_e/(m_i + m_e) \sim m_e$.

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