Order and Turbulence in rf-Driven Josephson Junction Series Arrays

Daniel Domínguez and Hilda A. Cerdeira*
International Centre for Theoretical Physics, P.O. Box 586, Miramare, 34100 Trieste, Italy
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We study underdamped Josephson junction series arrays that are globally coupled through a resistive shunting load and driven by an rf bias current. We find coherent, ordered, partially ordered, and turbulent regimes in the I-V characteristics. The ordered regime corresponds to giant Shapiro steps. In the turbulent regime there is a saturation of the broadband noise for a large number of junctions. This corresponds to a breaking of the law of large numbers already seen in globally coupled maps. Coexisting with this, we find an emergence of novel pseudosteps in the I-V characteristics.

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The dynamics of rf-driven Josephson junction arrays has been of great interest in recent years, both experimentally [1] and theoretically [2]. Much of the interest has concentrated in the study of giant Shapiro steps in two-dimensional arrays [1,2]. Also one-dimensional series arrays, but with a dc current drive, have been extensively studied when the junctions are globally coupled through an external shunting load [3,4]. Some investigations of chaos and turbulence on two-dimensional Josephson junction arrays, but where there is a locally coupled dynamics, have also been done recently [5]. Apart from being a realization of nonlinear dynamical systems with many degrees of freedom, Josephson junction arrays are devices that have potential applications as high frequency coherent power sources [6,7], parametric amplifiers, and voltage standards [6].

Here we study one-dimensional Josephson junction series arrays (JJSA) when they are driven by an *rf bias current*. It has been shown that underdamped, rf-driven, single Josephson junctions show chaotic behavior [8,9]. When these junctions are globally coupled in a JJSA, two conflicting trends will be present: destruction of coherence due to the chaotic divergences of the individual junctions, and synchronization through the global averaging of the common shunting load. This interplay between temporal chaos and space synchronization has been studied recently in globally coupled logistic maps (GCM) [10–13]. These systems exhibit coherent, ordered, partially ordered, and turbulent phases [10]. In particular, a surprising result was found by Kaneko [11]: in the turbulent phase, where spatial coherence is completely destroyed, a subtle collective behavior emerges. This was seen as a violation of the law of large numbers as a function of the number of logistic maps. In this paper, we show that the same kinds of phenomena exist in rf-driven underdamped JJSA. Moreover, we find that whenever the JJSA shows a breaking of the law of large numbers, novel pseudo-Shapiro steps emerge in the I-V characteristics of the JJSA. This last effect is a new result which does not result directly from the previously known phenomena in GCM.

We consider an underdamped JJSA shunted by a resistive load, and subjected to an rf bias current $I_B(t) = I_{\rm dc}+I_{\rm rf}\sin(\omega_{\rm rf}t)$ [14]. The dynamical behavior of Josephson junctions is commonly described with the resistively shunted junction model [15]. With this model, the governing equations of the JJSA [3] are

$$\ddot{\phi}_k + g\dot{\phi}_k + \sin\phi_k + i_L = i_{\rm dc} + i_{\rm rf}\sin(\Omega_{\rm rf}\tau) , \qquad (1)$$

$$i_L = \sigma v(\tau) = \frac{\sigma}{N}\sum_{j=1}^{N} g\dot{\phi}_j , \qquad (2)$$

where ϕ_k is the superconducting phase difference across the junction k, and $k = 1, \ldots, N$. We use reduced units, with currents normalized by the critical current, $i = I/I_c$; time normalized by the plasma frequency $\omega_p t = \tau$, with $\omega_p = \sqrt{2eI_c/\hbar C}$ and C the capacitance of the junctions; and voltages by rI_c, with r the shunt resistance of the junctions. Here, i_L is the current flowing through the resistive load; $g = (\frac{\hbar}{2eCr^2I_c})^{1/2} = 1/\beta_c^{1/2}$, with β_c the McCumber parameter [15]; $v = V_{\rm total}/N$ is the total voltage across the array per junction; $\sigma = \frac{rN}{R}$, with R the resistance of the shunting load, represents the strength of the global coupling in the array; and the normalized rf frequency is $\Omega_{\rm rf} = \omega_{\rm rf}/\omega_p$. Equation (1) represents current conservation, and Eq. (2) comes from requiring the total voltage across the array equal to the voltage across the load.

The simplest attractor of the system is the coherent state for which $\phi_k(\tau) = \phi_j(\tau) = \phi_0(\tau)$. In this case the equations reduce to

$$\ddot{\phi}_0 + \tilde{g}\dot{\phi}_0 + \sin\phi_0 = i_{\rm dc} + i_{\rm rf}\sin(\Omega_{\rm rf}\tau) , \qquad (3)$$

with $\tilde{g} = g(1+\sigma)$. This corresponds to the dynamics of one single Josephson junction. It is known that it can have chaotic behavior in the underdamped regime, i.e., for $\tilde{g} < 2$, and below the plasma frequency, $\Omega_{\rm rf} < 1$ [9]. In this paper we choose $\tilde{g} = 0.2$, $\Omega_{\rm rf} = 0.8$, and $i_{\rm rf} = 0.61$, and we analyze the behavior of the JJSA as a function of $i_{\rm dc}$, the coupling σ, and the number of junctions, N. We work with fixed $\tilde{g}$, instead of g, in order to have the same coherent attractor in all the cases . We integrate the dynamical equations using a fourth order Runge-Kutta method with fixed step $\Delta\tau = T/160$, with $T = 2\pi/\Omega_{\rm rf}$ the period of the rf drive, and we iterate the dynamics for times as long as $1024T$, after discarding the first 256 periods. For some particular cases, we have checked our

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results with $\Delta\tau = T/320$ and integration time $2048T$. For each run we used different sets of random initial conditions $\{\phi_k(0), \dot{\phi}_k(0)\}$.

One of the responses that can be measured experimentally is the I-V characteristics of the JJSA, which is the time average voltage per junction $\langle v(\tau)\rangle = \bar{v}$ as a function of i_{dc}. When the junctions are rf biased, they show Shapiro steps [8,9,16]. These are regions for which the average voltage is constant and $\bar{v} = \frac{n}{m} g\Omega_{rf}$. They correspond to phase locked states, which are periodic solutions in resonance with the rf current, either harmonic ($m = 1$) or subharmonic ($m > 1$). In other parts of the I-V it is possible to have chaotic solutions, in which the junction switches pseudorandomly between unstable, overlapping Shapiro steps [8,9]. We study the chaotic nature of the solutions by computing the maximum Liapunov exponent λ of the JJSA. Experimentally, most chaotic modes can be observed as broadband noise in the power spectrum of the voltage [8,9]. The power spectrum is computed as $S(\omega) = \frac{2}{T_m}|\int_0^{T_m} v(\tau)e^{i\omega\tau}d\tau|^2$. In the presence of broadband noise, the low frequency part of the spectrum approaches a constant, $S_0 = \lim_{\omega\to 0} S(\omega)$.

We first analyze the dynamics of one single Josephson junction with the parameters specified above. In Figs. 1(a) and 1(b) we show the I-V characteristics and Liapunov exponent, respectively. We have also computed S_0 (not shown), which essentially correlates with the behavior of λ in this case. We distinguish four different regimes as a function of i_{dc}. (i) There are periodic solutions, with $\lambda < 0$ and $S_0 \to 0$. They appear either below the critical current ($i_{dc} < i_c = 0.036$), where there is no average dissipation $\bar{v} = 0$, or at the Shapiro steps, which in this case are at voltages $\frac{1}{2}g\Omega_{rf}$ ($0.256 < i_{dc} < 0.428$) and $3g\Omega_{rf}$ ($0.476 < i_{dc} < 0.508$). (ii) There are chaotic solutions in the region between i_c and the step at $\frac{1}{2}g\Omega_{rf}$ ($0.036 < i_{dc} < 0.256$), for which $\lambda > 0$, S_0 finite. In this region some periodic "windows" are also seen (notably for voltages $\frac{1}{2}g\Omega_{rf}$ and $\frac{1}{3}g\Omega_{rf}$). (iii) For high currents ($i_{dc} > 0.508$), where there is a linear resistive behavior in the I-V, we find quasiperiodic solutions (also subharmonics with high m are possible here), for which $\lambda \approx 0$, $S_0 \to 0$. (iv) Finally, between the two steps, there is a region ($0.428 < i_{dc} < 0.476$) where either periodic solutions with $\bar{v} = \frac{1}{2}g\Omega_{rf}$, quasiperiodic solutions, or chaotic solutions can exist, depending on the initial conditions. In this region the I-V shows hysteresis. Note that we have deliberately chosen a case with few stable Shapiro steps. For this set of parameters, most of the Shapiro steps are unstable and overlapping, giving place to a wide region of chaotic states.

Now we study the spatiotemporal behavior of JJSA. Also in Figs. 1(a) and 1(b) we show the I-V curve and maximum Liapunov exponent for an array with 128 junctions and coupling $\sigma = 0.2$. With regard to the temporal behavior, we see two main differences with respect to the single junction. The chaotic region (ii) above $i_c = 0.03$ is narrower ($0.03 < i_{dc} < 0.2$), leaving place to periodic solutions corresponding to the Shapiro step at $\bar{v} = \frac{1}{2}g\Omega_{rf}$ ($0.2 < i_{dc} < 0.364$). On the other hand, the region (iv) with hysteresis is wider ($0.364 < i_{dc} < 0.514$), and shows more chaotic solutions than in the single junction case. This region has grown at the expense of part of the $\bar{v} = \frac{1}{2}g\Omega_{rf}$ Shapiro step and the $\bar{v} = 3g\Omega_{rf}$ Shapiro step. We find that this tendency is increased as a function of increasing σ, with the chaotic region (ii) narrowing and the region (iv) expanding in their respective ranges in i_{dc}.

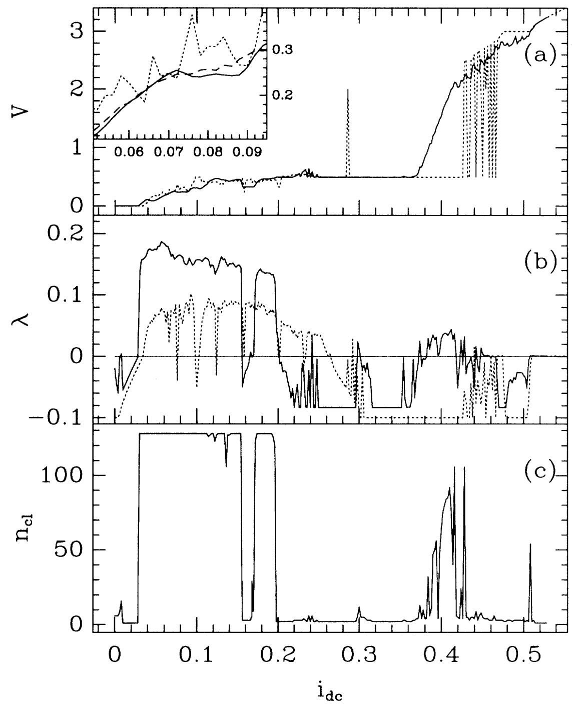

FIG. 1. (a) I-V characteristics for one single Josephson junction with $g = 0.2$, $\Omega_{rf} = 0.8$, $i_{rf} = 0.61$ (dotted line); and for a series array with the same parameters and $N = 128$ junctions with coupling $\sigma = 0.2$ (full line). We have normalized the average voltage as $V = \bar{v}/g\Omega_{rf}$. The inset is a blowup of the I-V curve in the region of low currents, showing the emergence of a pseudostep with increasing N ($N = 1$, dotted line; $N = 16$, dashed line; $N = 128$, full line). (b) Maximum Liapunov exponent λ as a function of i_{dc}. (c) Number of clusters n_{cl} as a function of i_{dc} for the series array with $N = 128$.

To further characterize these regimes in the JJSA, we analyze their spatial behavior. One important concept in globally coupled maps is "clustering" [10]. A cluster is defined as $\phi_i(t) = \phi_j(t)$ for i, j in the same cluster. An attractor can be characterized by the number of clusters it has, n_{cl}, and the number of elements of each cluster $(M_1, M_2, \ldots, M_{n_{cl}})$. For example, the coherent state is a one-cluster attractor ($n_{cl} = 1, M_1 = N$). In Fig. 1(c) we show n_{cl} as a function of i_{dc}, also for $N = 128$, $\sigma = 0.2$. We find different phases, according to their spatial behavior, which are as follows. (a) First, we find that the coherent attractor only exists either for currents below the critical current ($i_{\rm dc} < i_c = 0.3$, temporally periodic) or for high currents in the resistive regime ($i_{\rm dc} > 0.514$), corresponding to the temporally quasiperiodic region (iii). (b) In the Shapiro step at $\bar{v} = \frac{1}{2}g\Omega_{\rm rf}$ ($0.2 < i_{\rm dc} < 0.364$) there are few clusters, $n_{\rm cl} \ll N$, a behavior that corresponds to the "ordered" phase of GCM [10]. Here, for most of the currents is $n_{\rm cl} = 2$, and in the places where $n_{\rm cl} > 2$ (but $n_{\rm cl} \ll N$) almost all the junctions oscillate in two big clusters ($M_1 \approx N/2, M_2 \approx N/2, M_3 = 1, \ldots, M_{n_{\rm cl}} = 1$). (c) The temporally chaotic region (ii), $0.03 < i_{\rm dc} < 0.2$, has all the phases different, $n_{\rm cl} \sim N$, a behavior that corresponds to the termed "turbulent" phase of GCM [10]. There is also an "ordered" window with $\bar{v} = \frac{1}{3}g\Omega_{\rm rf}$ in the middle of the turbulent phase ($0.156 < i_{\rm dc} < 0.170$), for which $n_{\rm cl} = 3$. In fact, for the different cases we have studied, the ordered phase of JJSA seems to coincide with the Shapiro steps, with the number of big clusters being equal to the order m of the step. (d) The current range above the step of $\frac{1}{2}g\Omega_{\rm rf}$ that corresponds to the region (iv), $0.364 < i_{\rm dc} < 0.514$, even when it can have some temporally chaotic solutions, is clearly different from the turbulent phase in its spatial behavior. It can have (depending on the initial conditions) either attractors with few clusters, $n_{\rm cl} \ll N$, or attractors with many clusters, $n_{\rm cl} \sim N$, but with almost all the junctions concentrated in one or two of these clusters. This regime corresponds to the "partially ordered" or "glassy" phase of GCM [10]. We also find that while λ and S_0 change smoothly as a function of $i_{\rm dc}$ in the turbulent phase, they change wildly in the partially ordered phase.

How does the behavior of the JJSA depend on a function of N? We find that the turbulent phase is the one that shows the most notable changes with increasing N. In fact, we find a nonstatistical behavior for large N, like the one found by Kaneko in GCM [11] as a breaking of the law of large numbers. First of all, let us note that the voltage per junction $v^{(N)}(t) = \frac{1}{N}\sum_{j=1}^{N} g\dot{\phi}_j$ acts as a "mean field" in Eqs. (1) and (2). Since in the turbulent phase the $\phi_j(t)$, and therefore the $\dot{\phi}_j(t)$, take random values almost independently, one might expect that $v(t)$ will behave as an average noise. The power spectrum of $v(t)$ will be

$$S(\omega) = \frac{1}{N}|v_j(\omega)|^2 + \frac{1}{N^2}\left[\sum_{i\neq j} v_i(\omega)v_j^*(\omega)\right] , \qquad (4)$$

with $v_j(\omega)$ the Fourier transform of $v_j(t) = g\dot{\phi}_j(t)$. If the $\dot{\phi}_j(t)$ are completly independent, the second term in (4) will vanish for low frequencies, $\omega \to 0$. Therefore $S_0^{(N)} \sim \frac{1}{N}S_0^{(1)}$, with $S_0^{(N)}$ the low frequency part of the power spectrum of a JJSA with N junctions. This is the equivalent of the law of large numbers for a periodically driven system. Then we might expect that in the large N limit the the broadband noise part of $v^{(N)}(t)$ will tend to vanish ($S_0 \to 0$, for $N \to \infty$), reducing the dynamics of the JJSA to N independent chaotic junctions with an additional time periodic driving $v^{(N\to\infty)}(t)$.

In Fig. 2 we show the calculated values of S_0 as a function of N for different values of σ and for $i_{\rm dc} = 0.124$ (similar behavior is also seen for other values of $i_{\rm dc}$ within the turbulent phase). We see that for some values of σ, S_0 follows a $1/N$ behavior. But for some other values of σ, S_0 saturates for large N, indicating that some "order" has emerged in the turbulent phase. This corresponds to the breaking of the law of large numbers found in GCM [11,12]. This also affects the full power spectrum $S(\omega)$, where broad peaks develop for large N in the GCM [11,12]. We have seen the same behavior in the JJSA for the power spectrum of $v(t)$ [17].

We find that this subtle coherence of the turbulent phase notably affects the I-V characteristics of the JSSA in an unexpected way. We find that novel "pseudosteps" emerge in the I-V curve for large N at the same time that S_0 saturates in the turbulent phase. This is detailed in the inset of Fig. 1(a). There we see that, while for $N = 1$ the I-V curve in this region has a "noisy" aspect, when increasing N some pseudosteps tend to appear. Many pseudosteps are present all along the range of $i_{\rm dc}$ corresponding to the turbulent phase, as we show in Fig. 3(a) for $\sigma = 0.4$, $N = 128$. Note that $N = 128$ is a value before the full saturation of S_0, since it is hard to simulate very large N for the full I-V. However, we see that the pseudosteps emerge and sharpen up with increasing N, always in coexistence with a saturation of S_0. On the other hand, in Fig. 3(b), we show the case for $\sigma = 0.1$, for which we do not see a breaking of the law of large numbers. There is no evidence of any pseudosteps in the

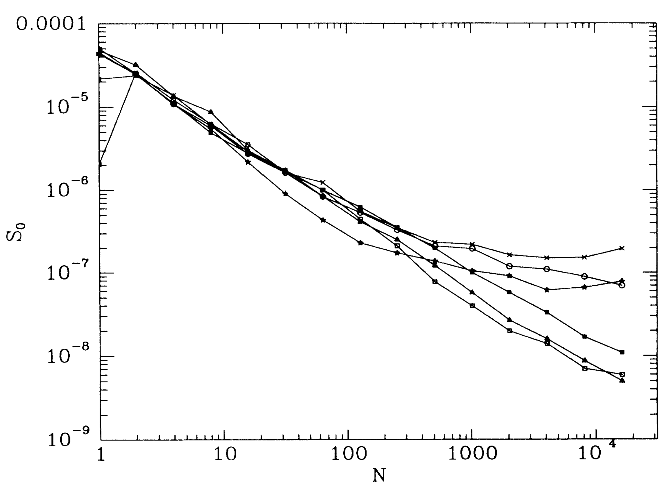

FIG. 2. Low frequency limit of the power spectrum, $S_0 = \lim_{\omega\to 0} S(\omega)$, as a function of the size of the array N, for $\tilde{g} = 0.2$, $\Omega_{\rm rf} = 0.8$, $i_{\rm rf} = 0.61$, $i_{\rm dc} = 0.124$ and different values of σ. (△, $\sigma = 0.1$; □, $\sigma = 0.15$; ⋆, $\sigma = 0.2$; ○, $\sigma = 0.3$; ×, $\sigma = 0.4$; ■, $\sigma = 0.5$.)

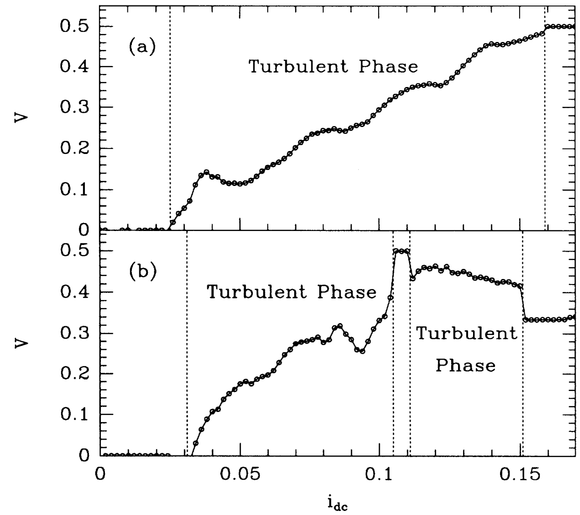

FIG. 3. I-V characteristics for an array with $N = 128$ junctions and with $\tilde{g} = 0.2$, $\Omega_{\rm rf} = 0.8$, $i_{\rm rf} = 0.61$. (a) $\sigma = 0.4$; note the development of pseudosteps; (b) $\sigma = 0.1$. The limits of the turbulent phase ($\lambda > 0$, $n_{\rm cl} \approx N$) are detailed.

I-V either.

The pseudosteps are not true Shapiro steps, since they do not correspond to mode locked periodic states. Instead, they have a positive Liapunov exponent and finite broadband noise emission. This emergence of pseudosteps within the turbulent regime of the JJSA is a new result which one could not have predicted from our previous knowledge of GCM. They seem to arise as an additional effect originated by the fact that we have a system of coupled nonlinear differential equations with a time periodic drive, instead of simply coupled logistic maps.

In conclusion, we find that many phenomena studied in GCM [10–13] can be measured in the laboratory in rf-driven JJSA through their I-V characteristics and power spectra. Charge density waves can be another candidate real system, where there are also many coupled degrees of freedom [18]. Also there, mode locking phenomena and Shapiro steps have been studied in large N systems [18], but so far few studies of chaos have been conducted in this case. Apart from finding an experimental realization of the breaking of the law of large numbers in the turbulent regime of the JJSA, we found that a new collective phenomenon coexists with it. This is the appearance of pseudosteps in the I-V characteristics. They can be experimentally distinguished from the true Shapiro steps since they only exist for large N and have a finite broadband noise S_0. Instead, the true Shapiro steps exist for any N and have $S_0 = 0$.

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* Also at Universidade Estadual de Campinas, Instituto de Fisica, 13081 Campinas, Sao Paulo, Brazil.

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